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ON SOME FUNDAMENTAL FLAWS IN PRESENT AEROACOUSTIC THEORY

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A critical analysis of the main theoretical approaches to the theory of aerodynamic sound is presented for the case of inviscid gas flow. Various systems of acoustic equations are considered, with and without externally assigned source terms. Decomposition of flow variables into acoustic and non-acoustic components is discussed as a complex problem which is crucial for studying many aeroacoustic phenomena, and a new concept is suggested for this problem. A time-averaging procedure, which is usually applied for such a decomposition, is examined, and its essential flaws are revealed. The general formulation of Lighthill's acoustic analogy is analyzed as well as various subsequent approaches to the theory of sound generation. This analysis shows that in all these the definitions of aerodynamic sound sources are featured by evident defects, and so these models cannot be adopted as physically or mathematically accurate. The relevant problems of experimental and computational research are considered in connection with the theoretical models under discussion.

1. INTRODUCTION

Undoubtedly, the distinguished textbook "*Theory of sound*" [1] by Lord Rayleigh remains as the true basis of acoustics. Many fundamental ideas have been expressed therein, and from time to time new facets of this celebrated scientific work, that give great impulse to further research efforts, have been revealed.

After World War II ended, the outstanding advances in aviation technology required a new level of comprehension in flow acoustics. Blokhintsev's textbook "*Acoustics of a nonhomogeneous moving medium*" [2] issued in 1946 must be emphasized as a significant stride in aeroacoustic theory. In fact, just that monograph opened the present half-century of intense research in aeroacoustics.

Presently, aeroacoustics can be viewed as flourishing, because again it is strongly demanded as a tool for numerous practical problems. Among those the problem of noise reduction is of particular interest. As an impressive example, the powerful turbofans for civil aircrafts could be mentioned. The noise produced by new counter-rotation propeller systems, as well as by helicopter propellers, poses other difficult questions. All these problems become crucial due to increasing activity in aerotransportation.

Usually, one follows two ways to reduce the total noise which is emitted by a jet engine or by any other unit with high-unsteady flow. Following the *first* way, one can change the geometry and structure of duct elements or apply some active devices, often with the aim of acting on the non-linear processes of hydrodynamic instability, and in turn weakening

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radically the total intensity of sound sources. If the *second* approach is taken, the generated sound could be suppressed by acoustically treating the duct, although there is little chance to do this in an optimal manner and without an essential economic penalty. The first way, which can be attributed to the topical problem of flow control with the use of innovative smart technologies, is the most difficult, but the most promising as well, and so it remains as the main challenge for the future since the current level of theoretical comprehension of fundamental aeroacoustic phenomena is evidently insufficient.

Sound waves are usually characterized by relatively small amplitudes in comparison with the background-flow variables, and so one should treat all the relevant problems with extreme accuracy. Indeed, any error in a theoretical or computational approach is easily able to exceed the magnitude of an acoustic disturbance, and then no valid solution can be obtained. Unfortunately, rather coarse approximations are often applied in aeroacoustics.

A number of recent papers (see references [3–44]), both theoretical and computational, could be cited to show that much active research is being conducted in the topical areas of aeroacoustics, and a sizable part of these efforts is related to the problems of sound generation in various flows. These works are, however, based on a very small number of the well-known theoretical approaches which are modified, simplified, and adapted to the particular flow conditions (mainly the most popular versions of Lighthill's acoustic analogy [45–51] including the Ffowcs-Williams and Hawkings equation [52], as well as diverse forms of Kirchhoff's theorem [53–58]). The key mathematical models of sound generation are usually adopted without any critical analysis, and such an absolute trust in those (though without absolute grounds for this) may lead to a certain stagnation in this research direction.

Today one can notice the lack of both new fresh ideas and impartial reviews of the current situation in this important scientific field. Some try to justify this by saying that generally the end of the 20th century is marked by a pronounced deceleration in the rate of scientific discoveries. Moreover, assertions can be found that all new research efforts will not change essentially the previously accumulated bulk of knowledge, at least not the theoretical fundamentals. For instance, such a concept has been expressed by Crocker in his article "The End of Science?" [59]: "In acoustics and vibration, perhaps most of the basic theoretical work has been done in many important areas and there is little need to change or expand these theoretical analyses". However, the total collection of theoretical fundamentals in aeroacoustics is evidently overestimated thereby, and all the general progress in fluid mechanics gradually gives rise to many doubts about the validity of conventional aeroacoustic theories. Naturally, expression of such doubts may discredit the habitual methods among which the Lighthill acoustic analogy is still recognized by many as the most general, accurate, and universal model for sound generation. For example, some of these methods were much praised in a few lectures on aeroacoustics ("Aeroacoustics of vorticity" by D. G. Crighton, "Acoustics of unstable flows" by A. P. Dowling, and "Computational aeroacoustics" by S. K. Lele) presented at the XIXth International Congress on Theoretical and Applied Mechanics held in Kyoto, Japan, August 1996.

At the 2nd European Fluid Mechanics Conference (Warsaw, Poland, September 1994) when giving a talk entitled "Simulation of non-linear acoustic phenomena in unsteady subsonic flows", the author presented a brief critical review of the current situation in theoretical aeroacoustics, and the main conclusion was as follows: "Thus, the main theoretical basis of aeroacoustics cannot be recognized as satisfactory. So far we have no unified system of equations which would govern adequately the nonlinear acoustic phenomena in high-unsteady flows, including the mechanism of sound generation. Obviously, this problem will not be solved without progress in the key question of how sound waves can be accurately separated out from a background unsteady flow, but this question is usually avoided in any discussions". That conclusion was irritating to some in the audience, but the author expressed similar negative opinions on the existing aeroacoustic theories while giving talks at various international conferences, having started this activity at the 13th International Congress on Acoustics (Belgrade, Jugoslavia, 1989). It is relevant to recall that paper [60] was the only one among all presented at the 5th ICSV held in Adelaide, Australia, December 1997, in which the fundamental theoretical aspects of aerodynamic sound sources were discussed, with sharp criticism of orthodox approaches. Nevertheless, in some lectures [36–40] presented at the 6th ICSV, held in Copenhagen, Denmark, July 1999, these approaches were offered again as the quite efficient ones, and any criticism of these was neglected.

Anyway, a half century of aeroacoustics has brought not only a lot of advances, mainly in practical fields, but a number of serious flaws, errors and delusions in the key theoretical models as well. One should recognize this before entering the second half of the aeroacoustic century. Perhaps, a realistic and impartial analysis of past experience will help to terminate the present state of euphoria in theoretical and computational aeroacoustics. In turn this may impel one to look more attentively at the next generation of theoretical models, such as those based on a radically new concept of aerodynamic sound sources [60–62].

So in this paper an attempt is made to destroy the myth that the present theoretical approaches in aeroacoustics are accompanied by a complete set of rigorous mathematical and physical proofs. It may look as if the author is merely trying to find the maximum number of defects in the existing theoretical models, but the major idea of this work is to attract attention to the true state of theoretical aeroacoustics, especially in the field of sound generation, so that an urgent need for revising the widely adopted fundamentals will become evident. In addition, this paper can be regarded as a necessary preface to the author's next paper in which a full description of a new non-linear theory of aerodynamic sound will be given.

The subsequent contents of this paper are as follows.

The closed system of dynamic and thermodynamic equations for an inviscid and non-heat-conducting fluid, subject to external mass addition, forcing and heat release, is presented in section 2. This mathematical model is sufficient for the purposes of this paper. Also, a set of the initial and boundary conditions is given to pose the relevant initial-boundary-value problem.

In section 3, Blokhintsev's linear equations [2] are analyzed for the case of inviscid gas flow without external sources. Despite evident ambiguities in defining both "acoustic field" and "quasisteady mean flow" within this model, one should agree with its concept that five scalar acoustic variables must be generally specified: namely the three acoustic components of the velocity (or momentum) vector, and two independent thermodynamic variables (from among, for instance, pressure, density, temperature, enthalpy, and entropy).

The general problems of decomposing an unsteady flow into an acoustic field and a background flow are discussed in section 4. The obtaining of two separate (although possibly interconnected, at least through new source terms) closed systems for these parts within a two-stage algorithm is regarded as the most correct way to an adequate theory of aerodynamic sound. The typical procedures of time averaging are considered, and a few parameters are there selected which are able to change radically the final result. As an example of a non-traditional approach, an unsteady irrotational homentropic flow is decomposed into an unsteady irrotational background flow and an irrotational acoustic field with distributed sound sources.

In section 5, various theoretical models are considered for the flows in which the externally assigned sources, mass and heat addition, as well as the imposed forces, may be present. In this case the key problem is to distinguish the source actions on the unsteady background flow and on the acoustic field. Note that this problem has not been solved within the well-known theoretical approaches.

In section 6, the striking conceptual defects of Lighthill's acoustic analogy are revealed, and so definite conclusions are given on its validity.

Section 7 shows the inherent flaws of various subsequent approaches to the theory of aerodynamic sound. These flaws may be partly explained by the fact that most of these approaches have arisen under the dominant influence of Lighthill's acoustic analogy.

In section 8, the extremely important role of theoretical fundamentals in experimental and computational study of aeroacoustic phenomena is emphasized. Obviously, one cannot do without an adequate aeroacoustic theory which should supplement these research methods.

The main conclusions are formulated in section 9.

2. BASIC EQUATIONS OF FLUID MECHANICS

It is sufficient here to consider only inviscid gas flows where the phenomena of sound generation and propagation are of particular interest. Therefore, the generation of vorticity by sound, primarily due to near-wall viscous effects, is beyond this analysis. The acoustics of viscous heat-conducting flows will be the subject of a future separate paper. The particular cases of acoustic waves with extremely high frequencies (ultrasound) and very low ones (infrasound) are excluded from the phenomena under study. Also, the supersonic flows, in which shock waves take place with a discontinuous entropy field, are not considered.

So the following basic system for inviscid gas flow is taken:

$$\partial\rho\mathbf{u}/\partial t + \nabla(\rho\mathbf{u};\mathbf{u}) + \nabla p = \mathbf{F} + \mathbf{k}, \tag{1}$$

$$\partial\rho/\partial t + \nabla(\rho\mathbf{u}) = \mu, \tag{2}$$

$$\partial s/\partial t + \mathbf{u}\nabla s = q, \tag{3}$$

$$\Im(s, p, \rho) = 0, \tag{4}$$

Here $\nabla(\rho\mathbf{u};\mathbf{u}) = (\rho\mathbf{u}, \nabla)\mathbf{u} + \mathbf{u}\nabla(\rho\mathbf{u})$, $\mathbf{u} = \{u_1, u_2, u_3\}$ is the flow velocity, p is the static pressure, ρ is the density, s is the entropy per unit mass, $\mathbf{F}$ is the external body force, μ is the mass source, and q is the entropy source per unit mass due to both volume heat release and a non-zero mass source. Vector $\mathbf{k}$ denotes the rate of momentum change because of mass sources.

Generally, $\mathbf{F}$, μ and q are supposed to be assigned functions of $\mathbf{r}$, t, p, ρ, i.e., the values of these source functions at each point of the flow domain do not depend explicitly on the quantity $\mathbf{u}$, and so these values remain invariant in any reference frame, including those which may arise after making Galilean transformations of spatial co-ordinates. By the way, the term "Galilean invariance" will be further used to imply that a certain function or a differential expression does not change its value in any inertial reference frame. As a simple example, the partial time derivative $\partial s/\partial t$ changes its value after Galilean transformation, but the expression $\mathrm{d}s/\mathrm{d}t = \partial s/\partial t + \mathbf{u}\nabla s$ retains its original magnitude, and so it is clear that both left- and right-hand side of equation (3) must be Galilean invariant. This feature seems to be very important in aeroacoustics since the reference frame may not be automatically connected with non-moving rigid walls of a duct or with a body, and the choice of an appropriate *local* reference frame may become most ambiguous when the flow regions are analyzed in which unstable jets generate sound.

Instead of equation (3) one can take the equation which governs the entropy balance in unit volume:

$$\partial \rho s/\partial t + \nabla(\rho s \mathbf{u}) = \mu s + \rho q.$$

The particular case is possible when $\mu \neq 0$ but $q = 0$, if the added fluid particles possess the local specific entropy of the mean flow. When $\mu = 0$, one can write the relation $q = Q_h (\rho T)^{-1}$ where Q_h is the heat release per unit volume, and T is the temperature.

One can assume that new fluid particles, arising due to mass sources, are at rest relative to the mean flow, and then $\mathbf{k} = \mu \mathbf{u}$. Besides, those particles do not change the local value of s if $q = 0$ (this implies that the mass sources produce the fluid particles which have the local specific entropy of the mean flow). Thereby the mass sources are here assumed to be convected by flow like small rigid particles in a surrounding gaseous medium. Formally, the case $\mathbf{k} = 0$ could be also considered (this would mean the non-moving mass sources which give zero total momentum change), but it is hard to imagine such a system of fixed mass sources continuously distributed in a gas flow. We discuss in detail these important questions because in the following we will compare some approaches where the wrong introduction of external sources has led to serious mistakes in the general conclusions.

After such a definition of vector $\mathbf{k}$, equation (1) can be replaced by

$$\partial \mathbf{u}/\partial t + (\mathbf{u}, \nabla)\mathbf{u} + (\nabla p)/\rho = \mathbf{f}, \quad \mathbf{f} = \mathbf{F}/\rho.$$

The medium will, to fix ideas, be regarded as a perfect gas: i.e., equation (4) is taken as

$$s = c_v \ln(p/\rho^\gamma), \quad \gamma = c_p/c_v = \text{const}, \quad c_v = R/(\gamma - 1), \quad p = R\rho T, \quad h = c_p T,$$

where h is the specific enthalpy, R is the gas constant, and c_p and c_v are the values of specific heats at constant pressure and constant volume respectively.

A certain initial-boundary-value problem can be posed for $\mathbf{Z}(\mathbf{r}, t) = \{Z_j\} = \{u_1, u_2, u_3, s, p, \rho\}$ $\mathbf{r} \in G$, $t \in J_t$ when system (1)–(4) is supplemented by the initial distributions $\mathbf{Z}(\mathbf{r}, 0)$ in G, as well as by the local boundary conditions (e.g. in accordance with the general approach [63]):

$$\Phi(u_n, p, \rho, \mathbf{r}_b, t) = 0 \quad \text{for any sign of } u_f \text{ if } M_n < 1 \tag{5}$$

$$s = \theta(\mathbf{r}_b, t) \quad \text{only if } u_f < 0. \tag{6}$$

Here Φ and θ are the assigned functions, $\mathbf{r}_b \in \Gamma$, $t \in J_t$ (we assume that $(\partial\Phi/\partial u_n)^2 + (\partial\Phi/\partial p)^2 \neq 0$), $u_n = \mathbf{un}$, with $\mathbf{n}$ the outward normal to the smooth boundary surface Γ, $u_f = u_n - u_b$, with u_b the assigned velocity of surface Γ along $\mathbf{n}$, and $M_n = |u_n|/a$. Condition (6) means that the entropy should be prescribed only for inflowing fluid particles. The normal velocity may be specified, instead of equation (5):

$$u_n = \phi(\mathbf{r}_b, t), \quad \mathbf{r}_b \in \Gamma, \; t \in J_t, \quad M_n < 1.$$

When $M_n > 1$ and $u_f < 0$, all variables must be assigned at the boundary.

In aeroacoustics particular attention should be paid to the boundary conditions. Even small changes in these may be decisive for the phenomena inside G , and so this gives a powerful tool of flow control. The boundary conditions on permeable “artificial” surfaces like the inflow and outflow sections of a duct are of extreme importance [64, 65]. If one applies the “traditional” set of boundary conditions at these surfaces while solving a computational problem, this may result in intense sound sources due to transformation of vortex disturbances into acoustic waves (see references [66, 67]), and this spurious effect may even exceed the true sound generation in all the internal volume.

A confined spatial domain, which represents only a small part of all the flow region under study, is often taken for the computational simulation. A more sizable domain may demand too many grid points, and in turn excessive computer time. Since the boundary conditions contain all the information about the outer space, the principle of uncertainty exists: one can diminish the size of the computational domain only by applying more complex boundary conditions, and sometimes these latter have to be most sophisticated. For instance, the application of *non-local* boundary conditions (some of them were suggested by the author in papers [63–68]), which contain complex procedures of active control over the processes in the whole domain, or at least near the boundary, may represent the necessary condition for accurate resolution of aeroacoustic phenomena.

3. LINEAR BLOKHINTSEV'S MODEL WITHOUT SOURCE TERMS

One can now consider again "the most general linear equations of flow acoustics" derived by Blokhintsev [2] for the analysis of small fluctuations in a "quasisteady" subsonic background flow. Suppose now that these equations are derived from the basic system (1)–(4) where $\mu = 0$, $q = 0$, $\mathbf{F} = 0$. It should be noted that in reference [2] the following source terms have been taken: $\mu = 0$, $q = 0$, $\mathbf{F} = \rho\mathbf{g}$, where $\mathbf{g}$ is the constant vector of gravitational acceleration, but here the problem of externally assigned sources is so important that it will be considered separately in section 5.

In reference [2] it was assumed that $\omega\tau \gg 1$ (τ is the characteristic time during which substantial changes in the mean-flow structure occur, ω is the sound frequency). This assumption is too indefinite to serve as an accurate condition of the model validity, although, as minimum, it might imply that the model was not intended to simulate the phenomena of sound generation by flow. In the following it will be shown that this model is by no means applicable to the investigation of acoustic processes in a high-unsteady flow.

Within the approach of reference [2] all variables were represented as

$$\mathbf{Z}(\mathbf{r}, t) = \mathbf{Z}_m(\mathbf{r}, t) + \mathbf{Z}_\varepsilon(\mathbf{r}, t), \quad \mathbf{r} \in G\,,\ t \in J_t, \tag{7}$$

where the small disturbances $\mathbf{Z}_\varepsilon = \{u_{1\varepsilon}, u_{2\varepsilon}, u_{3\varepsilon}, p_\varepsilon, s_\varepsilon\}$ are labelled by index "ε". It was also suggested that the mean flow with variables $\mathbf{Z}_m(\mathbf{r}, t) = \langle \mathbf{Z}(\mathbf{r}, t) \rangle_t$ (without giving any definite idea about what kind of averaging procedure should be applied) had to be described by a system which was exactly identical to the basic one (1)–(4). So the evident flaw of such an approach can be detected: the system governing $\mathbf{Z}_m(\mathbf{r}, t)$ is formally sufficient to simulate all kinds of waves, including sound. But if an "unsteady mean flow" is assumed to be governed by the most general system like equations (1)–(4), then one does not need any additional system for the simulation of small fluctuations. However, it seems to be highly improbable that the mean-flow system can acquire the absolutely same form as the basic system (1)–(4) after any procedure of time averaging has been applied to the latter, and so this approach may be most correctly applied only to the *steady* mean flows (i.e., $\mathbf{Z}_m(\mathbf{r}, t) \to \mathbf{Z}_0(\mathbf{r})$ when $\omega\tau \to \infty$). Hence, one must define accurately the averaging procedure, which having been applied to the original system (1)–(4), would yield the appropriate systems both for background flow and small fluctuations within a certain decomposition, but even in these fluctuations the sound waves may be inseparable. This delicate question was not discussed in reference [2], and so it will be considered in detail in section 4.

According to the method given in reference [2], one should substitute equations (7) into system (1)–(4) and omit the small terms of second order. As a result, the following closed linear system was derived to describe the evolution of small fluctuations

$$\partial \mathbf{u}_\varepsilon/\partial t + (\mathbf{u}_0, \nabla)\mathbf{u}_\varepsilon + (\mathbf{u}_\varepsilon, \nabla)\mathbf{u}_0 = (\rho_\varepsilon \nabla p_0)/\rho_0^2 - (\nabla p_\varepsilon)/\rho_0, \tag{8}$$

$$\partial \rho_\varepsilon/\partial t + \nabla(\rho_0 \mathbf{u}_\varepsilon + \rho_\varepsilon \mathbf{u}_0) = 0, \tag{9}$$

$$\partial s_\varepsilon/\partial t + \mathbf{u}_0 \nabla s_\varepsilon + \mathbf{u}_\varepsilon \nabla s_0 = 0, \tag{10}$$

$$p_\varepsilon = s_\varepsilon(\partial p_0/\partial s_0)_\rho + \rho_\varepsilon(\partial p_0/\partial \rho_0)_s. \tag{11}$$

The following important peculiarity has been noted in reference [2]: the right-hand side of equation (8) cannot be represented as a gradient of a certain function if $\nabla s \neq 0$, and so generally curl $\mathbf{u}_\varepsilon \neq 0$ even in a quiescent medium. Hence, one can introduce the potential φ_ε (while $\mathbf{u}_\varepsilon = \nabla\varphi_\varepsilon$) only in a limited number of particular cases.

This system shows that in a flow with a non-homogeneous entropy field one cannot do without considering the entropy disturbances. So it has been clearly pointed out that generally one needs to use all the five independent scalar variables $\{u_{1\varepsilon}, u_{2\varepsilon}, u_{3\varepsilon}, p_\varepsilon, s_\varepsilon\}$ to specify the field of disturbances $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$. Of course, in comparison with the classical acoustics of homentropic gaseous media this looks rather unaccustomed. Nevertheless, this concept did not represent a radical extension of the previous approaches. For instance, in reference [1] the acoustic problems were considered in which one had to deal with the complete set of fluctuation variables $\{u_{1\varepsilon}, u_{2\varepsilon}, u_{3\varepsilon}, p_\varepsilon, s_\varepsilon\}$, at least while analyzing the wave processes in a viscous, heat-conducting gas.

The following linear equation has been also derived in reference [2] for the particular case of irrotational disturbances in the irrotational homentropic steady mean flow:

$$\frac{\mathrm{d}^2\varphi_\varepsilon}{\mathrm{d}t^2} - a_0^2 \Delta\varphi_\varepsilon - (\nabla h_0, \nabla\varphi_\varepsilon) - \frac{\mathrm{d}\varphi_\varepsilon}{\mathrm{d}t}(\mathbf{u}_0, \nabla \ln a_0^2) = 0. \tag{12}$$

Here $\varphi = \varphi_0(\mathbf{r}) + \varphi_\varepsilon(\mathbf{r}, t)$, $\mathbf{u}_0 = \nabla\varphi_0$, $\mathbf{u}_\varepsilon = \nabla\varphi_\varepsilon$, $\mathrm{d}\varphi_\varepsilon/\mathrm{d}t = \partial\varphi_\varepsilon/\partial t + \mathbf{u}_0\nabla\varphi_\varepsilon = -p_\varepsilon/\rho_0$; variables φ_0, h_0 and a_0 represent the potential, enthalpy and sound speed in a steady mean flow which is assumed to be known. This equation is quite unique in that the disturbance field is now described by the single scalar variable φ_ε. Probably, the potential φ_ε may be decomposed as a sum $\varphi_\varepsilon = \varphi_\alpha + \varphi_v$ where φ_α and φ_v correspond to the acoustic and non-acoustic components respectively; however, generally it is not trivial to find a rigorous procedure for such a decomposition, and perhaps the latter can be implemented in different ways. Anyhow, it is clear that unsteady irrotational flow of compressible fluid is not composed solely of sound waves. Surely, the non-acoustic components will not contribute directly to the far sound field if an infinite domain G_∞ is considered, but their connection with possible sound sources in a finite domain G, or in the local flow region $G_f \subset G_\infty$, remains ambiguous (see also section 4.4), much less if a quite accurate theory for aerodynamic sound sources has not been applied to this problem yet.

Thus, in the author's opinion, system (8)–(11), where no clear separation between sound waves and non-acoustic disturbances has been made, cannot be classified as "the most general system of *acoustic* equations". In fact, this model may be regarded only as a kind, or even a certain extension, of the conventional approach which was often used before in numerous problems when the hydrodynamic stability of steady flows was analyzed. Nevertheless, since the disturbances $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ include sound waves, it is quite possible that in some particular cases this system of equations may be helpful in studying sound propagation phenomena.

4. ON THE GENERAL PROBLEM OF FLOW DECOMPOSITION

4.1. PROCEDURES OF TIME-AVERAGING

The problem of defining the mean flow (or background flow) is very important in aeroacoustics. One usually applies a certain averaging procedure to the basic system of non-linear evolutionary equations with the aim to obtain a new linearized system for the evolution of small disturbances. For instance, consider the conventional method when the integral operator of time averaging is applied to a function $\mathbf{Z}(\mathbf{r}, t)$ which is defined in a spatial domain G and within a time interval $J_t = (0, t_f)$:

$$\mathbf{Z}_m(\mathbf{r}, t) = \langle \mathbf{Z} \rangle_t = \frac{1}{\delta t} \int_{t_1}^{t_2} \mathbf{Z}(\mathbf{r}, t')\, \mathrm{d}t', \tag{13}$$

$$\delta t = \tau_1 + \tau_2, \quad t_1 = t - \tau_1, \quad t_2 = t + \tau_2, \quad \tau_1 \geqslant 0, \quad \tau_2 \geqslant 0, \quad t_1 \in J_t, \quad t_2 \in J_t, \quad \mathbf{r} \in G.$$

Here the dependence of $\langle \mathbf{Z} \rangle_t$ on t is emphasized, because generally t_1 and t_2 are functions of t within J_t (moreover, τ_1 and τ_2 may be complex functions of t). Evidently, the result depends greatly on the form of the function $\mathbf{Z}$ as well as on the interval of averaging δt one has chosen. Then the function $\mathbf{Z}$ can be decomposed as

$$\mathbf{Z}(\mathbf{r}, t) = \mathbf{Z}_m(\mathbf{r}, t) + \mathbf{Z}_\varepsilon(\mathbf{r}, t). \tag{14}$$

By the way, the following relation can be obtained from equation (14):

$$\mathbf{Z}_m(r, t) = \langle \mathbf{Z}_m(\mathbf{r}, t) \rangle_t + \langle \mathbf{Z}_\varepsilon(\mathbf{r}, t) \rangle_t;$$

this, however, gives one no reason to demand that $\langle \mathbf{Z}_\varepsilon(\mathbf{r}, t) \rangle_t \equiv 0$ or $\langle \mathbf{Z}_m(\mathbf{r}, t) \rangle_t \equiv \mathbf{Z}_m(\mathbf{r}, t)$.

As a simple example, suppose that

$$Z(\mathbf{r}, t) = f(\mathbf{r}) \sin \omega t, \quad \tau_1 = \tau_2 = \tau = \text{const}, \quad \omega = \text{const}, \quad t \in (-\infty, \ \infty).$$

Then the functions

$$\langle Z \rangle_t = \frac{1}{2\tau} \int_{t-\tau}^{t+\tau} Z(\mathbf{r}, t')\, \mathrm{d}t' = (\omega\tau)^{-1} (\sin \omega t)(\sin \omega\tau) f,$$

$$Z_\varepsilon = Z - \langle Z \rangle_t = [1 - (\omega\tau)^{-1} \sin \omega\tau] f \sin \omega t$$

depend not only on both $\mathbf{r}$ and t, but on τ as well, and these three cases illustrate that

1. $\langle Z \rangle_t \to Z(\mathbf{r}, t) = f \sin \omega t, \quad Z_\varepsilon \to 0$ when $\tau \to 0$;
2. $\langle Z \rangle_t = (-1)^n (\omega\tau)^{-1} f \sin \omega t$, $Z_\varepsilon = [1 - (-1)^n (\omega\tau)^{-1}] f \sin \omega t$, $\langle Z_\varepsilon \rangle_t = (-1)^n (\omega\tau)^{-1} [1 - (-1)^n (\omega\tau)^{-1}] f \sin \omega t$ when $\omega\tau = n\pi + \pi/2$, $n = 0, 1, 2, \ldots$;
3. $\langle Z \rangle_t = 0$, $Z_\varepsilon = Z = f \sin \omega t$, $\langle Z_\varepsilon \rangle_t = 0$ when $\omega\tau = n\pi$, $n = 1, 2, \ldots$.

If different characteristic frequencies are present in Z, for instance when

$$Z(\mathbf{r}, t) = \sum_{k=1}^{N} f_k(\mathbf{r}) \sin(k\omega_0 t), \quad \tau_1 = \tau_2 = \tau = \text{const},$$

additional difficulties arise in choosing the optimal value of τ. Surely, this simple function can be averaged by taking $\tau = n\pi/\omega_0$, $n = 1, 2, \ldots$, and then for all N this case will be similar to the third one from those given above. However, if diverse non-multiple frequencies are present in the spectrum, the problem of choosing the minimal τ may become extremely complicated.

If one takes a certain function $\mathbf{Z}(\mathbf{r}, t)$, which is known in $G \times J_t$, and J_t is a finite interval, the procedure of time averaging can be formally implemented by choosing $t_1(t)$ and $t_2(t)$ in different ways, but in all cases one must meet the necessary demand: $t_1 \in J_t$, $t_2 \in J_t$. This means that $\tau_1 \to 0$ when $t \to 0$, as well as $\tau_2 \to 0$ when $t \to t_f$. Besides, the equality $\tau_1 = \tau_2$ may be desirable for better approximation. For instance, one could take $\tau_1 = \tau_2 = t$ when $0 < t < t_f/2$, and $\tau_1 = \tau_2 = t_f - t$ when $t_f/2 < t < t_f$.

If one tries to apply this procedure to an *unknown* function $\mathbf{Z}(\mathbf{r}, t)$ (i.e. that function is to be obtained as a solution of a definite initial-boundary-value problem which is posed in $G \times J_t$), then one may be unable to choose the key parameters τ_1 and τ_2 in the manner that would provide $\langle Z_\varepsilon \rangle_t = 0$. Also, any possible procedure of time averaging will be closely connected with the whole method applied to the solution of that problem. If one uses a finite-difference scheme, which implies step-by-step advancement in time from the assigned initial distribution $\mathbf{Z}(\mathbf{r}, 0)$, it seems illogical to apply procedure (13) where $\tau_2 > 0$.

Anyway, one can try to apply a certain procedure of time averaging to the non-linear equations of basic system (1)–(4). If one has obtained a solution $\mathbf{Z}(\mathbf{r}, t)$ of an initial-boundary-value problem posed in $G \times J_t$, both the mean-flow variable $\mathbf{Z}_m(\mathbf{r}, t)$ and disturbances $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ can be readily determined as well if functions τ_1 and τ_2 are specified. But usually such a procedure is applied to system (1)–(4) before any solution is obtained, and then the problem of flow decomposition may become very complicated. Indeed, instead of five independent scalar variables $\{Z_j\} = \{u_1, u_2, u_3, p, \rho\}$ one will deal with 10 unknown variables: $\{Z_{mj}\}$ and $\{Z_{\varepsilon j}\}$.

Suppose that the basic non-linear system (1)–(4) can be rewritten in the compact form

$$\partial \mathbf{Z}/\partial t + L_k(\partial \mathbf{Z}/\partial x_k) = \mathbf{Y}, \quad \mathbf{Z} = \{Z_j\}, \quad j = 1, \ldots, 5, \quad k = 1, 2, 3, \tag{15}$$

where matrices $L_k = L_k$ $(\mathbf{Z})$, and the source $\mathbf{Y} = \mathbf{Y}(Z) = \{\mathbf{f}, \mu, q\}$. After applying a decomposition procedure like that defined by equation (14) one must forget the old variable $\mathbf{Z}$, and only new variables $\mathbf{Z}_m = \{Z_{mj}\}$ and $\mathbf{Z}_\varepsilon = \{Z_{\varepsilon j}\}$ are to be used. Then equation (15) should be rewritten as

$$\partial \mathbf{Z}_m/\partial t + \partial \mathbf{Z}_\varepsilon/\partial t + L_k \, \partial(\mathbf{Z}_m + \mathbf{Z}_\varepsilon)/\partial x_k = \mathbf{Y}, \tag{16}$$

where $L_k = L_k(\mathbf{Z}_m + \mathbf{Z}_\varepsilon)$, $\mathbf{Y} = \mathbf{Y}(\mathbf{Z}_m + \mathbf{Z}_\varepsilon)$. A curious fact should be noted: system (16), if considered separately, in its appearance does not display any details of the averaging procedure which has been applied for the flow decomposition. At first sight, this seems unimportant, and this illusion may be the reason why many authors did not give due attention to the decisive peculiarities in procedure (13). One should be aware that now a transition has been implemented from the *local* system of partial differential equations (15) to the *non-local* system of integro-differential equations (16), and the accurately specified formula (13) represents the only exact link between $\mathbf{Z}$ and $\mathbf{Z}_m$ that depends greatly on both $\tau_1(t)$ and $\tau_2(t)$.

Presumably, the whole problem of flow decomposition could be solved in the following way. First one should find a solution for the mean-flow variable $\mathbf{Z}_m(\mathbf{r}, t)$ which is assumed to be independent of $\mathbf{Z}_\varepsilon$. But this can be done only if one succeeds in deriving a separate closed system for $\mathbf{Z}_m$. At the second stage one would obtain $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ by taking $\mathbf{Z}_m(\mathbf{r}, t)$ as a known function. Again, it should be emphasized that variable $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ may include all kinds of disturbances but not solely sound waves. Following this idea, one can group the terms in equation (16) as

$$\partial \mathbf{Z}_m/\partial t + \partial \mathbf{Z}_\varepsilon/\partial t + \mathbf{D}_m + \mathbf{D}_\varepsilon + \mathbf{D}_{m\varepsilon} = \mathbf{Y}_m + \mathbf{Y}_\varepsilon, \tag{17}$$

where $\mathbf{D}_m$ represents the non-linear expression which contains $\mathbf{Z}_m$ and $\partial\mathbf{Z}_m/\partial x_k$, $\mathbf{D}_\varepsilon$ depends on $\mathbf{Z}_\varepsilon$ and $\partial\mathbf{Z}_\varepsilon/\partial x_k$, but $\mathbf{D}_{m\varepsilon}$ includes both $\mathbf{Z}_m$ and $\mathbf{Z}_\varepsilon$ as well as their spatial derivatives (i.e., $\mathbf{Z}_m$ and $\mathbf{Z}_\varepsilon$ are "inseparable" in $\mathbf{D}_{m\varepsilon}$), $\mathbf{Y}_\varepsilon = \mathbf{Y}(\mathbf{Z}_m + \mathbf{Z}_\varepsilon) - \mathbf{Y}(\mathbf{Z}_m)$. Then equation (17) can be rewritten as

$$\partial\mathbf{Z}_\varepsilon/\partial t + \mathbf{D}_\varepsilon + \mathbf{D}_{m\varepsilon} - \mathbf{Y}_\varepsilon = \mathbf{Q}_m \quad \text{where } \mathbf{Q}_m = -\partial\mathbf{Z}_m/\partial t - \mathbf{D}_m + \mathbf{Y}_m. \tag{18}$$

Since the variable $\mathbf{Z}_m$ is supposed to be known within the separate problem for $\mathbf{Z}_\varepsilon$, the vector $\mathbf{Q}_m$ may be regarded as an externally assigned source term which is independent of $\mathbf{Z}_\varepsilon$. What should be particularly emphasized is that a certain "source term" like $\mathbf{Q}_m$ will appear after any procedure of decomposition one applies, with time averaging or without (see also section 4.3), and just this represents a general approach (although it may be implemented in different ways) to the definition of sources determined by the evolution of mean flow $\mathbf{Z}_m(\mathbf{r}, t)$.

Unfortunately, the key question remains unsolved whether a closed system for $\mathbf{Z}_m$ can be composed in a proper manner. Of course, in efforts to do this one can try to use another system,

$$\langle\partial\mathbf{Z}_m/\partial t\rangle_t + \langle\partial\mathbf{Z}_\varepsilon/\partial t\rangle_t + \langle\mathbf{D}_m\rangle_t + \langle\mathbf{D}_\varepsilon\rangle_t + \langle\mathbf{D}_{m\varepsilon}\rangle_t = \langle\mathbf{Y}_m\rangle_t + \langle\mathbf{Y}_\varepsilon\rangle_t, \tag{19}$$

which is obtained by applying the integral operator of time averaging to equation (17). But system (19) is unable to yield the exact relations between the new variable $\mathbf{Z}_m$ and its temporal and spatial derivatives, because generally $\langle\partial\mathbf{Z}_m/\partial t\rangle_t \neq \partial\langle\mathbf{Z}_m\rangle_t/\partial t$, $\langle\partial\mathbf{Z}_\varepsilon/\partial t\rangle_t \neq \partial\langle\mathbf{Z}_\varepsilon\rangle_t/\partial t \neq 0$, $\langle Z_{mi}\partial Z_{mj}/\partial x_k\rangle_t \neq \langle Z_{mi}\rangle_t\partial\langle Z_{mj}\rangle_t/\partial x_k$, and so on. To simplify this problem, the following approximate rules are often used:

$$\langle Z_1 + Z_2\rangle_t \approx \langle Z_1\rangle_t + \langle Z_2\rangle_t, \quad \langle Z_1 Z_2\rangle_t \approx \langle Z_1\rangle_t\langle Z_2\rangle_t + \langle Z_{1\varepsilon}Z_{2\varepsilon}\rangle_t,$$

$$\langle\langle Z_1\langle Z_2\rangle_t\rangle_t \approx \langle Z_1\rangle_t\langle Z_2\rangle_t, \quad \langle\langle Z\rangle_t\rangle_t \approx \langle Z\rangle_t, \quad \langle\partial Z/\partial x_k\rangle_t \approx \partial\langle Z\rangle_t/\partial x_k,$$

$$\langle Z_\varepsilon\rangle_t \approx 0, \quad \langle Z_\varepsilon\langle Z\rangle_t\rangle_t \approx 0, \quad \langle\partial Z_\varepsilon/\partial x_k\rangle_t \approx 0, \quad \langle\partial Z_\varepsilon/\partial t\rangle_t \approx 0, \ldots,$$

although a lot of examples can be found when such relations cause excessive errors (one could consider again the simple function $Z(\mathbf{r}, t) = f(\mathbf{r})\sin\omega t$). Indeed, after applying these rules one would have $\langle\partial\mathbf{Z}_\varepsilon/\partial t\rangle_t = 0$ in equation (19), but all the same, terms like $\partial\langle Z_{\varepsilon i}Z_{\varepsilon j}\rangle_t/\partial x_k$ will remain. Hence, one should formulate a number of additional assumptions, perhaps rather coarse, to provide the closure. For instance, one could suppose that all the second-order terms like $\partial\langle Z_{i\varepsilon}Z_{j\varepsilon}\rangle_t/\partial x_k$ are negligible in system (19). Also, amid the most frequently applied approaches, the existence of an approximate steady solution $\mathbf{Z}_m(\mathbf{r})$ is assumed in some particular flows (this model will be considered below). But if one recalls the assumption given in reference [2] that the averaged system may have exactly the same form as the basic system (15), this idea cannot be accepted in general. This would correspond to the limiting case $\delta t \to 0$ when the averaged system tends to the basic system (15), and then it is able to describe all high-unsteady processes, including sound effects.

By the way, similar problems arise when one tries to apply the Reynolds equations for turbulent flows within the model of incompressible fluid where acoustics is beyond consideration. Then the additional multiparameter models of small-scale turbulence are usually designed with the use of experimental data in order to close the system. In various problems of hydrodynamic stability, where the evolution of small disturbances is studied, the simplest geometries are usually taken for the approximation of a steady mean flow, but *instability of unsteady background flow* is still a vague notion which may be clarified in future. Surely those problems depart substantially from the specific aeroacoustic problems, but the extreme intricacy of these latter are often underestimated.

Even if one has designed a certain closed system for $\mathbf{Z}_m(\mathbf{r}, t)$ by using a definite procedure of time averaging, and then a relevant solution for $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ has been obtained, this does not mean that the global aeroacoustic problem has been solved, because an additional set of difficult questions may arise in turn. For instance, the $\mathbf{Z}_\varepsilon$-system is able to govern sound waves with the frequencies $\omega > \omega_0$, but at the same time sound waves with $\omega < \omega_0$ may be well described by the evolutionary $\mathbf{Z}_m$-system. Probably, this effect can be corrected by changing the characteristic parameters in the procedure of time averaging. Anyhow a method should be found which will enable one to distinguish the acoustic and non-acoustic disturbances which may appear within both systems. Evidently, all these questions cannot be solved merely by making the time averaging in the "optimal manner". One should also remember that in real processes of sound generation the characteristic time scales for changes in both unsteady background flow and the sound field generated have the same order of magnitude. Moreover, the amplitudes of acoustic waves may be comparable with the amplitudes of non-acoustic fluctuations in the flow region $G_f \subset G$ where intense sound sources occur.

So it is very difficult to find the most universal and quite accurate method of specifying δt in a typical procedure of time averaging (13). Thus, it seems hardly probable that such a way is able to yield an accurate and rather simple system for the evolution of time-averaged unsteady background flow in aeroacoustics, except for a number of simple particular cases. Nevertheless, many recent works can be mentioned where the method of "short-period averaging" is used, even in the simulation of complex turbulent flows (see e.g. references [4,18]), though without comprehensive discussion of the above relevant questions. In some papers one can find assertions that the key parameter δt may be chosen "arbitrarily" (for instance, as a certain period of flow observation in experiment), and usually that implied no further discussion on this subject. However, in the above it has been clearly shown that such an arbitrariness may lead to unpredictable and unacceptable results in theoretical models.

4.2. LINEAR ACOUSTIC MODELS WITH TIME AVERAGING

Now one can recall the routine, but the most consistent procedure of time averaging which is widely applied in many problems of fluid mechanics and aeroacoustics [25, 69–73]: interval δt is there taken so large that all the mean-flow variables *are independent of time*. Often, to avoid any ambiguities, this interval is supposed to be *infinite* (surely then the interval J_t is assumed to be infinite as well). Then by decomposing

$$\mathbf{Z}(\mathbf{r}, t) = \mathbf{Z}_0(\mathbf{r}) + \mathbf{Z}_\varepsilon(\mathbf{r}, t), \quad \mathbf{r} \in G. \tag{20}$$

one can derive the linearized system for the evolution of small *unsteady* fluctuations $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ on the background of *steady mean flow* with variable $\mathbf{Z}_0(\mathbf{r})$ which is supposed to be the *known* steady solution of system (1)–(4) under the time-averaged set of boundary conditions. Actually, only a similar kind of time averaging could be implied in reference [2]. This approach may be inapplicable to the problems with finite time interval $J_t = (0, t_f)$, although formally one can take $t_1 = 0$, $t_2 = t_f$ for any $\mathbf{r}$ and t, so that it gives $\langle \mathbf{Z} \rangle_t = \mathbf{Z}_0(\mathbf{r})$.

The delicate question should be emphasized: generally one cannot prove either existence of a steady solution, or its uniqueness. In many real flows strong non-linear effects of hydrodynamic instability may develop after introducing a small disturbance. In fact, viscous flows at considerable Reynolds numbers are usually unsteady, and even at low

Reynolds numbers quite different steady solutions can be obtained under the same boundary conditions (e.g., subsonic flows in a plane duct with sudden symmetric expansion [74–76]). Thus, one cannot guarantee that a *unique* steady solution will be found, and a rough approximation of $\mathbf{Z}_0(\mathbf{r})$ may be unacceptable as well. Moreover, the following evident conradiction may arise: if all boundary conditions are independent of time, then decomposition (20) implies the existence of both steady and unsteady solutions under the fixed set of similarity criteria and boundary conditions.

Now it may look as if the obtaining of the most accurate solution $\mathbf{Z}_0(\mathbf{r})$ is the key problem in aeroacoustics. Not at all; usually no one tries to find an analytic approximation for $\mathbf{Z}_0(\mathbf{r})$ more complex than the near-parallel flow with uniform pressure field. This may be justified in that any function of $\mathbf{r}$, which has a distant likeness with the hypothetical mean flow, can be taken as $\mathbf{Z}_0(\mathbf{r})$. Formally, this will be valid if one substitutes the sum $\mathbf{Z} = \mathbf{Z}_0(\mathbf{r}) + \mathbf{Z}_\varepsilon(\mathbf{r}, t)$ into system (1)–(4) with the aim of finding a solution for the unknown function $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$. However, the following important aspect of such a primitive procedure should be pointed out: it is most probable that function $\mathbf{Z}_\varepsilon$ is too far from acoustics, and moreover, the ratio $\|\mathbf{Z}_\varepsilon\|/\|\mathbf{Z}_0\|$ may not be small.

But even if an appropriate approximation for $\mathbf{Z}_0(\mathbf{r})$ has been found, after such a decomposition the fluctuations contain all the information about the flow evolution, and so they describe simultaneously all kinds of waves: sound as well as the disturbances of both entropy and vorticity, but this cannot be recognized as an advantage.

Such a mean flow can, however, be regarded as steady only in the *unique* reference frame one has taken, and any Galilean transformation of co-ordinates will destroy the model. This is the serious flaw of this procedure.

A curious paradox should be mentioned. Suppose that one assumes the non-uniform mean flow to be stationary in a certain reference frame, and then uses system (8)–(11),written in that frame, for simulating the evolution of small fluctuations. But in any other reference frame, resulting from a Galilean transformation, the same mean flow should be recognized as unsteady, and then system (8)–(11) is unable to describe the small disturbances on a background of an *unsteady* flow. Hence, system (8)–(11) simulates all kinds of small disturbances only in the unique reference frame, but nobody has given an accurate procedure for choosing the best frame for any flow under study. Perhaps, a certain unsteady flow may be treated as steady after changing the reference frame in a proper manner. Generally, this problem is connected with the much more general question whether it is possible to introduce the invariant definition of *non-radiating flow with steady structure* (see reference [60]).

In analyzing this model it is very helpful to investigate the ways in which those disturbances can be generated in real flows. If the initial conditions at $t = 0$ imply a steady flow in a bounded spatial domain under study, any changes in the boundary conditions will result in the generation of all kinds of waves. Generally, it is hardly possible to arrange a device of boundary control so that it introduces solely acoustic disturbances. But even if one is able to generate only sound waves, they will cause gradually the appearance of vorticity perturbations at diverse boundary surfaces (primarily due to the effects of viscosity at the walls, especially near the sharp edges) which, being convected by shear flow, may give rise to the processes of hydrodynamic instability. These latter may lead to the effects of sound generation, and in turn new vorticity disturbances will appear, and so on. Besides, the external forcing terms (e.g., the gravity force) may be the reason for other instability effects. Anyhow, after a certain time one will have to consider sound waves on the background of a flow which is essentially unsteady.

Consequently, one comes to the undeniable conclusion that any time-averaging procedure formally applied to the evolutionary equations of fluid mechanics is not able to give a general and quite accurate way of separating out the net acoustic fluctuations, except in a few particular cases (uniform flow, the geometric acoustics approximation, etc.). Nevertheless, many attempts have been made to adapt this linear model for the solution of aeroacoustic problems. The thorough analysis of temporal and spatial scales in a definite flow, where some predictable features of both the mean-flow structure and the sound field could be taken into account, may enable one to separate out sound waves within this model. For instance, in subsonic flows the pronounced difference between the flow velocity and the speed of sound may be helpful in separating out sound waves, but in transonic flows this approach does not work.

Perhaps, the following general method could be applied to this type of problems. Since system (8)–(10) consists of homogeneous linear equations (without external sources), the unique general solution to an initial-boundary-value problem will represent a certain sum of partial solutions,

$$\mathbf{Z}_\varepsilon(\mathbf{r}, t) = \sum_{j=1}^{N} A_j \mathbf{Z}_{\varepsilon j}(\mathbf{r}, t),$$

where N may be infinite, and coefficients A_j are determined by the initial and boundary conditions. After obtaining the general solution one can analyze the summands $\mathbf{Z}_{\varepsilon j}$, where all kinds of disturbances are included with diverse characteristic frequencies and the speeds of propagation, with the aim of distinguishing the sound waves from the rest. In the case of an infinite spatial domain G_∞ Sommerfeld's radiation conditions can be applied in order to separate out only those solutions which contribute to the far sound field. Thus, any practical experience in the accurate analysis of this kind would be of much value. However, appraising the current state of this research direction, now one has to repeat the phrase from section II of reference [77]: "No solution to these equations subject to general initial and boundary conditions has been found for general mean flows".

Modern computational methods look developed enough to provide one with an adequate tool for the solution of even a three-dimensional non-linear version of systyem (8)–(11), but the practical realization of complex initial-boundary-value problems is often accompanied by numerous difficulties, which may be compared with those one encounters when applying the general system (1)–(4). For instance, a proper finite-difference scheme must resolve accurately the small-amplitude sound waves (in all directions of their propagation through subsonic flow) in a wide range of frequencies, as well as the disturbances of both entropy and vorticity convected by mean flow, but very few schemes are able to meet these stringent requirements.

Thus, only a number of specific problems have been solved with the use of simplified versions of system (8)–(11). As an example, to obtain solutions for the sound propagation in ducts [77, 78], researchers often restricted their attention to two flow configurations: parallel mean flows and near-parallel mean flows. Among others, the normal-mode method is frequently used when one is looking for an oscillating acoustic solution in a steady subsonic parallel mean flow on the basis of a linear *homogeneous* system (8)–(11) (i.e., without any external sources and forces). For two-dimensional ducts having uniform cross section, one can assume that

$$\mathbf{Z}_0(\mathbf{r}) = \{u_0 = u_0(y), \quad v_0 = 0, \quad T_0 = T_0(y), \quad p_0 = \text{const}\},$$

and the solution is supposed to have the form

$$Z_\varepsilon(\mathbf{r}, t) = \mathbf{W}E, \quad \mathbf{W} = \mathbf{W}(y), \quad E = \exp[\mathrm{i}(kx - \omega t)], \tag{21}$$

or

$$\mathbf{W} = \{u_\varepsilon = U(y)E, \quad v_\varepsilon = V(y)E, \quad p_\varepsilon = P(y)E, \quad \rho_\varepsilon = D(y)E\}$$

where the x-axis coincides with the duct axis ($x > 0$), and the y-axis is normal to it, k is the complex wavenumber, and ω is the dimensionless frequency. Amidst possible values of k and ω, determined from the relevant dispersion equation, one should select only those which are related to the sound propagation. In the particular mean flow with constant temperature this problem is reduced to the second-order equation for $P(y)$ obtained by Pridmore-Brown [79]. It should be noted that such solutions are independent of any initial conditions which may be specified in G at $t = 0$, and only boundary conditions at the inlet section $x = 0$ as well as on the walls of a duct are essential. Surely, by assuming the form of solution as expresion (21) or in some other way, one restricts oneself greatly, and so these solutions (there the forced response characteristics of the flow are usually found and not its intrinsic wave modes) represent only a small part amidst all the variety of possible solutions which could be obtained if quite general initial-boundary-value problems were posed in the same spatial domain. Moreover, it is absolutely impossible to study some important flow–sound interactions by this method. For instance, complex vortical transonic flow in a duct is able to distort substantially any initial sound field, and so the latter will become too far from the form given by equation (21). Nevertheless, this approximate approach, well described in numerous textbooks on linear acoustics, seems to be quite appropriate in a definite class of acoustic problems. By the way, a similar approach is often used in the study of hydrodynamic stability of parallel flows of incompressible fluid. Clearly, these solutions, as waves with definite sets $\{k, \omega\}$, may be also found (and then omitted as non-acoustic) amid the above solutions to the compressible fluid flows.

A number of comprehensive works have appeared recently [80–82, 34] in which more general approaches were suggested for the linear problems of wave propagation in ducts with steady mean flows. For instance, reference [82] includes the analysis of both acoustic and swirling modes under the influence of time-periodic body forces, and this may be a promising way to the separate study of different wave types within linear models.

So one should be very careful in applying any time-averaging procedure to the general equations of fluid mechanics, because it must be clearly understood which phenomena are to be described by the average variables and which others can be attributed to the fluctuations. In turn, some of these fluctuations can be approximated within a model of a near-incompressible medium, but others should be regarded as sound waves, and so on.

The author has gained considerable experience in solving diverse problems of fluid mechanics with the use of time-averaging procedures. For example, the computational simulation of unsteady subsonic turbulent flows was carried out [83]. There a version of the unsteady Reynolds equations for a compressible medium (clearly, those are based on the procedure of time averaging within the characteristic period $\delta t \gg l/u$, where l and u are the characteristic spatial scale and the average velocity of turbulent pulsations respectively), were used simultaneously with a multiparameter model of small-scale turbulence where the fluid was assumed as near-incompressible. The finite-difference scheme filtered all sound waves with length $\lambda \leqslant h$ (h is the maximum size of spatial grid), but the resulting evolutionary system was able to resolve the wide range of acoustic waves with $\lambda \gg h$ including those generated by large-scale vortices with the characteristic length $L \gg h$, $L \gg l$. Actually, the characteristic period τ_s of coherent sound waves generated by flow, as a period in the evolution of large vortex structures, should meet the demand $\tau_s \gg \delta t$. However, in some cases all these conditions were not satisfied simultaneously, and that led to substantial errors.

As an alternative to procedure (13), one can implement spatial averaging as

$$\langle \mathbf{Z}(\mathbf{r}, t)\rangle_s = \frac{1}{V_s}\int_{G_s} \mathbf{Z}(\mathbf{r}_s, t)\,\mathrm{d}\tau, \quad \mathbf{r}_s \in G_s\ |\mathbf{r}_s - \mathbf{r}| < r_G,\ G_s \subset G,$$

where G_s is a sphere (with radius r_G and volume V_s) centered at point $\mathbf{r}$ [70]. Clearly, the result will strongly depend on the radius of averaging r_G, and so one will meet with difficulties similar to those described above.

Probably, combined averaging in both time and space may lead to much more accurate solutions, although the difficulties in applying such a procedure may turn out to be unacceptable (the resulting integro-differential equations would be *non-local* in both time and space). For instance, analyzing only the local temporal changes in flow variables, one can find that the small-scale vorticity disturbances (assuming that a vortex of size L is convected by mean flow with the average velocity U) which are regarded as nearly incompressible, may produce the same characteristic time $\tau_s = O\ (L/U) = O\ (\lambda/a)$ in the temporal derivatives as the period of a low-frequency sound wave with length λ, although $L \ll \lambda$ may hold true. Hence, the additional analysis of spatial scales would be very desirable in such problems.

So while considering a particular flow one could estimate all temporal and spatial scales for both mean flow and diverse types of disturbances (e.g. in the course of the computational solution), and that would help one to determine the range where one's aeroacoustic model remains valid. But such a procedure does not seem to be the universal, and much less simple, remedy for accurately separating out the sound waves from those disturbances.

As a brief conclusion, one can dare to say that a linear system like system (8)–(11), which is based on the time-averaging procedure, cannot serve as a quite accurate theoretical model for the solution of general initial-boundary-value problems in aeroacoustics, because it describes all kinds of disturbances without explicit separation of sound waves. Nevertheless, this method is often applied to the simulation of sound propagation phenomena in a *quasisteady mean flow*, although a number of serious questions arise when one tries to define such a flow. If one analyzes a certain approach of this kind, a number of heuristic assumptions are usually brought up to predict the main spatial structure of the sound field (like in the routine case of parallel mean flow), but by doing this one oversimplifies in advance the prospective solution, and so the final result may be too far from reality.

4.3. OTHER WAYS TO DECOMPOSE THE FLOW

An alternative approach is known in aeroacoustics (see also section 6) when one tries to separate out the acoustic components in an unsteady flow by decomposing the velocity field as

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_p, \quad \operatorname{div} \mathbf{u}_s = 0, \quad \operatorname{curl} \mathbf{u}_p = 0. \tag{22}$$

Surely, in this way one can decompose any vector field, even without specifying the physical meaning of the variable $\mathbf{u}$. However, in the case of a finite spatial domain G the decomposition (22) can be implemented in a unique manner only if the normal velocity $\mathbf{nu}$ has been definitely decomposed at each point of the boundary surface, but generally this can be done in diverse ways. Doing this in an infinite domain, one should meet the specific demands to the behavior of $|\mathbf{u}|$ when $\mathbf{r} \to \infty$. For a general non-linear initial-boundary-value problem in fluid mechanics, it is highly improbable that one can find a non-conradictory method of creating an accurate two-medium model with two separate closed systems of equations for the variables $\mathbf{Z}_s(\mathbf{r}, t)$ and $\mathbf{Z}_p(\mathbf{r}, t)$ which would correspond to $\mathbf{u}_s$ and $\mathbf{u}_p$ respectively. This is very difficult since then one should decompose *in a unique manner* the velocity field as well as the distributions of both pressure and density. Besides, all the set of initial and boundary conditions must be split as well, and this separate problem is also far from simple. Anyway, the major question is usually omitted, that is whether $\mathbf{u}_p$ can be attributed solely to the sound field, and $\mathbf{u}_s$ to the background flow. Indeed, generally the background-flow velocity may be decomposed as a sum of potential and solenoidal components as in equations (22), and the acoustic velocity may be split in a similar manner. This method may be successful in a particular case only if all the attendant problems mentioned above are solved together, but usually some of them are ignored.

In section 5.2 of reference [72] a decomposition like equations (22) was applied to the simplest linear system of equations that governed the small disturbances $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ in a steady *uniform* homentropic flow ($\mathbf{u}_0 = \text{const}$). Then the fluctuating velocity $\mathbf{u}_\varepsilon$ was decomposed as $\mathbf{u}_\varepsilon = \mathbf{u}_p + \mathbf{u}_s$, where $\mathbf{u}_p$ and $\mathbf{u}_s$ were definitely treated as "acoustic oscillating velocity" and "vortical velocity" respectively, the pressure fluctuations being determined solely by $\mathbf{u}_p$. It seems unproductive to analyze this example, where the convected disturbances of vorticity and the sound waves propagate without any interaction, because it is too primitive to extend its conclusions on the great number of non-linear problems for vortical flows.

In references [84, 85] Doak proposed the original procedure of flow decomposition

$$\rho\mathbf{u} = \mathbf{b} - \mathbf{w}, \quad \nabla\mathbf{b} = 0, \quad \mathbf{w} = \nabla\psi,$$

where $\mathbf{b}$ is regarded as a "turbulent" component, and $\nabla\psi$ is assumed to be a sum of "acoustic" and "thermal" parts. These conceptual definitions, given as "the only possible", give rise to a number of questions. First of all, while considering the continuity equation, one can notice that expressions $\nabla\rho\mathbf{u}$ and $\partial\rho/\partial t$, being taken separately, are not Galilean invariant (in contrast to $\nabla\mathbf{u}$). Indeed, if $\{\nabla(\rho u)\}_0 = 0$ in the original inertial reference frame K_0, in a new frame K_m, which moves with the translational velocity $\mathbf{U} = \text{const}$ relative to K_0, for the same flow one will have

$$\{\mathbf{u}\}_m = \{\mathbf{u}\}_0 - \mathbf{U}, \quad \{\nabla(\rho\mathbf{u})\}_m = \{\mathbf{u}\}_m\nabla\rho + \rho\nabla\{\mathbf{u}\}_m = \{\nabla(\rho\mathbf{u})\}_0 - \mathbf{U}\nabla\rho = -\mathbf{U}\nabla\rho \neq 0.$$

This means that the above decomposition of $\rho\mathbf{u}$ can be implemented only in the "unique" reference frame; otherwise the potential ψ as well as the term $\Delta\psi$ must change depending on the choice of reference frame in order to retain the validity of the continuity equation $\partial\rho/\partial t - \Delta\psi = 0$.

So it seems that vector $\rho\mathbf{u}$ is not the best variable for such a procedure of decomposition. Besides, the time-averaging procedure was used to define the time-dependent fluctuations of all variables in a steady mean flow, but the main flaws of this approach have been discussed in section 4.2. These "acoustic" and "thermal" components of ψ' in the linear approximation are explicitly related to the fluctuations of pressure p' and entropy s' respectively, but evidently p' may also contain the non-acoustic part connected with the solenoidal velocity field, and as well the evolution of s' generally depends on both solenoidal mean flow and sound waves. Consequently, the way given in references [84, 85] is unlikely to be accepted as the most appropriate procedure of flow decomposition in aeroacoustics.

4.4. DECOMPOSITION OF IRROTATIONAL HOMENTROPIC FLOW

Finally, in this section the important particular case of irrotational homentropic flow is considered. Then the closed system of non-linear equations can be obtained from equations

(1)–(4) for the two scalar variables $\{\varphi, h\}$,

$$\partial\varphi/\partial t + (\nabla\varphi)^2/2 + h = H_0 = \text{const}, \tag{23}$$

$$\mathrm{d}h/\mathrm{d}t + a^2\Delta\varphi = 0, \tag{24}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{u}, \nabla)$, $\mathbf{u} = \nabla\varphi$, $a^2 = (\gamma - 1)h$. Also, the following non-linear equation (as a generalization of linear equation (12)) can be readily derived from this system:

$$\mathrm{d}/\mathrm{d}t\left(\frac{\partial\varphi}{\partial t} + \frac{(\nabla\varphi)^2}{2}\right) - a^2\Delta\varphi = 0, \tag{25}$$

Usually this equation is called an "exact non-linear equation governing the sound propagation in irrotational homentropic flow". However, this equation, being taken separately, is not closed (in contrast to equation (12)), and only the complete system (23)–(24) is able do govern the evolution of variables $\{\varphi, h\}$. What is also important is that the variables $\{\varphi, h\}$ may include both "acoustic" and "non-acoustic" components. So the problem of separating out the acoustic components from $\{\varphi, h\}$ remains like that which could be posed within linear equation (12). Besides, the fundamental conclusion (or more likely the fundamental delusion) has been universally adopted that any irrotational homentropic flow by no means can yield sound sources. This opinion may be partly explained by the fact that equation (25) resembles something like "an extended form of the sound propagation equation" without any terms which would be treated definitely as the sound sources. However, such a conclusion is influenced greatly by the way in which one defines the sound sources produced by a certain unsteady mean flow, and that way is closely connected with the general problem of flow decomposition into acoustic and non-acoustic components.

As an example of a non-traditional approach to this problem, a subsonic irrotational homentropic flow is analyzed in a finite spatial domain G. The boundary surface Γ_w may include movable or permeable parts, and even a rigid body (with rather smooth boundary Γ_β) may be immersed in that gas medium. The normal velocity $u_n = \mathbf{un} = \psi(\mathbf{r}_b, t)$, $\mathbf{r}_b \in \Gamma$, $\Gamma = \Gamma_\beta \cup \Gamma_w$ is everywhere assigned as the boundary condition, and for simplicity one can assume that

$$\int_\Gamma \psi\,\mathrm{d}\sigma = 0.$$

The initial conditions may be specified as $\mathbf{u}(\mathbf{r}, 0) = \mathbf{U}(\mathbf{r}) \neq 0$, $\operatorname{curl}\mathbf{U} = 0$, $\mathbf{nU} = \psi(\mathbf{r}_b, 0)$, $h(\mathbf{r}, 0) = h_0(\mathbf{r})$.

This flow is decomposed as

$$\varphi = \varphi_v + \varphi_\alpha, \quad \mathbf{Z} = \{\mathbf{u}, h\} = \mathbf{Z}_v + \mathbf{Z}_\alpha,$$

$$\mathbf{u} = \nabla\varphi, \quad \mathbf{u}_v = \nabla\varphi_v, \quad \mathbf{u}_\alpha = \nabla\varphi_\alpha, \quad \nabla\mathbf{u}_v = \Delta\varphi_v = 0,$$

where the subscripts v and α label the "background-flow variables" and the "irrotational disturbances" respectively. At the same time the boundary conditions are split as follows,

$$\mathbf{nu}_v = \mathbf{nu} = \psi(\mathbf{r}_b, t), \quad \mathbf{nu}_\alpha = 0, \quad \mathbf{r}_b \in \Gamma, \quad t \in J_t,$$

and thereby one has minimized $|\mathbf{nu}_\alpha|$ all over the boundary. The initial conditions are decomposed as $\mathbf{u}_v(\mathbf{r}, 0) = \mathbf{U}_v(\mathbf{r})$, $\nabla\mathbf{U}_v = 0$, $\mathbf{u}_\alpha(\mathbf{r}, 0) = \mathbf{U}_\alpha(\mathbf{r})$, $h_v = h_0(\mathbf{r})$, $h_\alpha = 0$; and further it will be assumed that $\mathbf{U}_\alpha = 0$.

Then at the first stage the following elliptic boundary-value problem is solved for the "unsteady background flow" $\mathbf{Z}_v$ (there all sound waves are characteristically precluded) at each moment $t \in J_t$,

$$\Delta\varphi_v = 0, \quad \partial\varphi_v/\partial n = \psi(\mathbf{r}_b, t), \tag{26}$$

and the enthalpy h_v will be determined from the additional equation

$$\partial\varphi_v/\partial t + \mathbf{u}_v^2/2 + h_v = H_0. \tag{27}$$

The great advantage of system (26), (27) is that one can use a lot of ready solutions obtained within the classical model of incompressible fluid flow.

Meanwhile, it should be emphasized that this system, if being considered separately, does not describe the steady flow of compressible fluid (where $\nabla\rho\mathbf{u} = 0$ rather than $\nabla\mathbf{u} = 0$) since equation (24) is not valid for $\mathbf{Z}_v$. Therefore, one cannot assign the initial conditions that would imply a steady mean flow along with $\mathbf{Z}_\alpha = 0$.

At the second stage, $\mathbf{Z}_v(\mathbf{r}, t) = \{\varphi_v, h_v\}$ is taken as a known function in $G \times J_t$, and then one can write the exact system of non-linear equations for $\mathbf{Z}_\alpha(\mathbf{r}, t) = \{\varphi_\alpha, h_\alpha\}$ as

$$\frac{\mathrm{d}_v}{\mathrm{d}t}\left(\frac{\mathrm{d}_v\varphi_\alpha}{\mathrm{d}t} + \frac{(\nabla\varphi_\alpha)^2}{2}\right) - \nabla\varphi_\alpha\nabla h - a^2\Delta\varphi_\alpha = Q_v, \tag{28}$$

$$\frac{\mathrm{d}_v\varphi_\alpha}{\mathrm{d}t} + \frac{(\nabla\varphi_\alpha)^2}{2} + h_\alpha = 0, \tag{29}$$

where

$$Q_v = -\frac{\mathrm{d}_v}{\mathrm{d}t}\left(\frac{\partial\varphi_v}{\partial t} + \frac{\mathbf{u}_v^2}{2}\right) = \frac{\mathrm{d}_v h_v}{\mathrm{d}t}, \quad \frac{\mathrm{d}_v}{\mathrm{d}t} = \frac{\partial}{\partial t} + (\mathbf{u}_v, \nabla), \quad a^2 = (\gamma - 1)(h_v + h_\alpha).$$

Now the invariant expression Q_v, which contains only the background-flow variables, may be regarded as a sound source for the field $\mathbf{Z}_\alpha$, although this is not quite accurate, and generally this decomposition does not provide one with rigorous proofs that $\mathbf{Z}_\alpha$ represents the sound field (e.g., decomposition of a steady flow may lead to $\mathbf{Z}_\alpha \neq 0$). This ambiguity is also shown by the fact that a steady background flow gives $Q_v = (\mathbf{u}_v, \nabla h_v) \neq 0$. Anyhow, $\mathbf{Z}_\alpha$ does contain a sound field, since the latter by no means can be included in $\mathbf{Z}_v$. What should be also noted is that $\mathbf{Z}_\alpha(\mathbf{r}, 0) = 0$ has been specified as well as $\mathbf{n}\mathbf{u}_\alpha = 0$ on the boundary, but this means that any sound effects may be initiated only by the volume source Q_v.

If one supposes the disturbances $\mathbf{Z}_\alpha$ to be small (at the same time some limitations should be imposed on the norm of Q_v), then, after omitting the quadratic terms, the linear system will be derived in the form

$$\frac{\mathrm{d}_v}{\mathrm{d}t}\left(\frac{\mathrm{d}_v^2\varphi_\alpha}{\mathrm{d}t}\right) - \nabla\varphi_\alpha\nabla h_v - a_v^2\Delta\varphi_\alpha = Q_v,$$

$$\frac{\mathrm{d}_v\varphi_\alpha}{\mathrm{d}t} + h_\alpha = 0,$$

and in the particular case of a quiescent medium, where $\mathbf{u}_v \equiv 0$, $Q_v \equiv 0$, and $\nabla h_v \equiv 0$, this system is reduced to the ordinary acoustic equations.

Also, one could pose the more general initial-boundary-value problem where

$$\int_{\Gamma} \psi \, \mathrm{d}\sigma = \eta(t) \neq 0.$$

Indeed, the non-local model [86] can be applied to the approximation of the unsteady subsonic background flow where the characteristic Mach number $M \ll 1$. Then equations (27) and (29) remain valid, but instead of equation (26) one should solve the other elliptic problem

$$\Delta\varphi_v = f(t) = \eta/V, \quad \partial\varphi_v/\partial n = \psi(\mathbf{r}_b, t),$$

where $V(t)$ is the volume of domain G. As a result, the following extended version of equation (28) is obtained

$$\frac{\mathrm{d}_v}{\mathrm{d}t}\left(\frac{\mathrm{d}_v\varphi_\alpha}{\mathrm{d}t} + \frac{(\nabla\varphi_\alpha)^2}{2}\right) - \nabla\varphi_\alpha \nabla h - a^2\Delta\varphi_\alpha - (\gamma-1) f h_\alpha = Q_v^*, \tag{30}$$

$$Q_v^* = \frac{\mathrm{d}_v h_v}{\mathrm{d}t} + (\gamma - 1) f h_v.$$

In any case one comes to a qualitative but extremely important conclusion: it seems quite possible to make a similar decomposition of an irrotational homentropic flow into the irrotational background flow and the irrotational field of disturbances, so that in the relevant equations (non-linear or linearized) for the "irrotational disturbances" a certain term may represent mainly the sound source.

A number of simple examples could be given to show this mechanism of sound generation in irrotational flows. For instance, consider the plane homentropic irrotational flow in a fixed cylindrical domain with a moving rigid cylindrical body inside it (the similar flow of an incompressible fluid was given in section 6.53 of reference [87]). The initial positions of both cylinders at $t = 0$ are exactly symmetric (see Figure 1(a)), and the instantaneous picture of background flow at $t = t_1 > 0$ is shown in Figure 1(b). The initial flow field at $t = 0$, where $\nabla\rho\mathbf{u} = \nabla\mathbf{u} = 0$ as well as curl $\mathbf{u} = 0$, is specified as follows:

$$u_r = 0, \quad u_\theta = k/r, \quad h = h_1 + \frac{k^2(r^2 - r_1^2)}{2r_1^2 r^2}, \quad k = \text{const}, \quad h_1 = \text{const},$$

and this field is decomposed as $\mathbf{Z}(\mathbf{r}, 0) = \{\mathbf{u}, h\} = \mathbf{Z}_v(\mathbf{r}, 0), \mathbf{Z}_\alpha(\mathbf{r}, 0) = 0$.

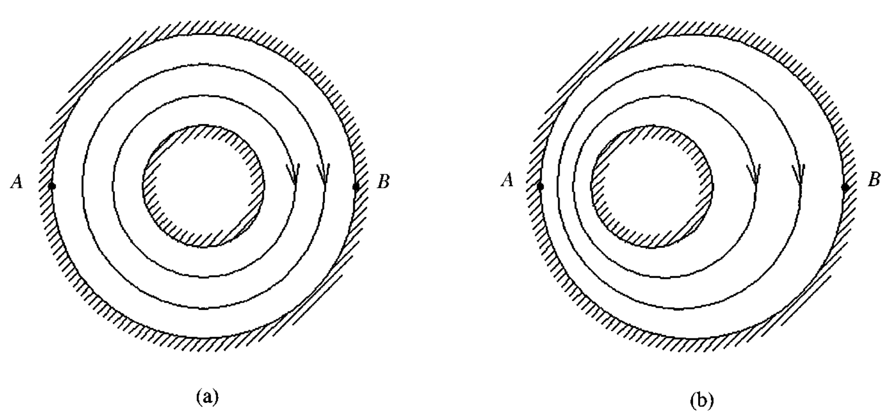

Figure 1. Two instantaneous pictures of the background flow: (a) $t = 0$; (b) $t = t_1 > 0$.

Consider two characteristic points in the domain: A and B. For those one can write

$$u_{vx}(A, t_1) = u_{vx}(B, t_1) = 0,$$

$$u_{vy}(A, t_1) > u_{vy}(A, 0),\ u_{vy}(B, t_1) < u_{vy}(B, 0),$$

$$h_v(A, t_1) < h_v(A, 0),\ h_v(B, t_1) > h_v(B, 0),$$

and so within the time interval $(0, t_1)$ one has positive and negative values of the volume source $Q_v \approx \partial h_v/\partial t$ near points B and A respectively (there $\mathbf{u}_v \nabla h_v \approx 0$ is assumed). Clearly, this source distribution causes acoustic oscillations all over domain G.

Of course, one could simulate such flows by using system (23), (24) without any decomposition. Then no explicit expression would appear for a volume source, although the whole process would be the same (the absence of volume sources would be compensated by the non-zero values of **nu** on the surface of internal cylinder in the above example). But this flow has been decomposed just with the aim of estimating the values of volume sound sources from the previously found solution to the rather simple elliptic problem posed for *unsteady background flow*. This two-stage concept seems most promising in the study of sound generation phenomena, at least in subsonic flows, although it has been radically modified in the approach [61]. Generally, in efforts to find an adequate definition of aerodynamic sound sources one cannot dispense with a decomposition of flow variables into "acoustic" and "background-flow" components (see also section 7). Anyway, the above analysis looks quite sufficient to refute the widely adopted delusion that an irrotational homentropic flow is unable to yield sound sources.

4.5. ON THE DEFINITION OF ACOUSTIC DISTURBANCES IN UNSTEADY FLOW

Before proceeding to further sections, it seems very helpful to recall again the key question in aeroacoustics: what is the difference between "acoustics" and "non-acoustic motion" in the general case? Though this question has been discussed from time to time, and moreover, the increasing activity in the simulation of turbulent compressible flows has given new impulse to considering this question, no radical progress has resulted from those discussions. Probably, the extreme complexity of the general non-linear system (1)–(4), which was taken as basic in the relevant theoretical research, represented the main reason for those unsuccessful attempts. Hence, one might feel some doubts that an adequate procedure could be correctly defined for separating out the sound disturbances from all five independent scalar variables $\{u_1, u_2, u_3, s, p\}$ of a high-unsteady flow. Nevertheless, a general solution to this non-linear problem has been first found by the author (that was briefly presented in references [60–62]), and its comprehensive description will be given in further papers. So the author's point of view on this fundamental problem can be briefly expounded now, at least in qualitative terms.

First of all, one has to remember that the general system of non-linear equations governing an inviscid gas flow (1)–(4) is classified as *hyperbolic in time and space*. However, among these hyperbolic properties one should distinguish two types: the *first* type, related to the convection of fluid particles by flow, is featured by characteristics $\mathrm{d}x_j/\mathrm{d}t = u_j$; the *second* one with two sets of characteristics $\mathrm{d}x_j/\mathrm{d}t = u_j + a$ and $\mathrm{d}x_j/\mathrm{d}t = u_j - a$ should be attributed to the propagation of sound waves (i.e., the longitudinal waves due to compressibility of a fluid). The parabolic processes due to viscosity and heat conductivity are excluded from this consideration.

As was discussed above, it is very difficult to find an adequate procedure of flow decomposition in aeroacoustics. It has been shown that a time-averaging procedure, for instance with a finite interval δt, being applied to all components of vector $\mathbf{Z}(\mathbf{r}, t)$ does not guarantee that one will derive an accurate closed system for the mean-flow variable $\mathbf{Z}_m(\mathbf{r}, t)$. But even if one succeeds in deriving such a system, it will lead to the filtering of sound waves with lengths $\lambda \leqslant a\,\delta t$, although the "long sound waves" with $\lambda \gg a\,\delta t$ may still exist. In the case $\delta t = \infty$, no sound waves propagate within a system written for *steady* mean flow, but then one has to deal again with the problem of separating out the sound waves from all kinds of disturbances which are present in a system like system (8)–(11), and this "new" problem is not easier than a similar one which could be posed within system (1)–(4).

Following a new two-stage concept [60–62] the flow variable $\mathbf{Z}(\mathbf{r}, t)$ must be represented as a sum $\mathbf{Z} = \mathbf{Z}_v + \mathbf{Z}_\alpha$. Then the basic system (1)–(4) as well as all the set of initial and boundary conditions should be accurately decomposed into two separate initial-boundary-value problems posed within two *closed* non-linear systems derived for the unsteady background flow with vector variable $\mathbf{Z}_v(\mathbf{r}, t)$ and for the acoustic field component $\mathbf{Z}_\alpha(\mathbf{r}, t) = \{u_{1\alpha}, u_{2\alpha}, u_{3\alpha}, s_\alpha, p_\alpha\}$. Surely these two problems are connected, due in part to the new source terms such as those in equation (18). Within this concept the unsteady background flow should correspond to the "*globally compressible*" fluid medium in which all sound waves are characteristically excluded (i.e., this implies the infinite speed of sound propagation), but all the rest of the dynamic processes can be simulated. Note also the following striking fact: the usual form of the equation of state $\Im(s_v, p_v, \rho_v) = 0$ is valid in this medium, and the formally calculated value $a_v^2 = \gamma p_v/\rho_v$ is finite.

Then in the particular case of subsonic flow the $\mathbf{Z}_v$-system will display the following local characteristic properties in time and space: partly elliptic (due to the infinite speed of sound) as well as partly hyperbolic with characteristics $\mathrm{d}x_j/\mathrm{d}t = u_j$ which reflect only non-acoustic effects. Thereby, these properties resemble those one can find in the classical model of incompressible fluid flow.

New unusual properties can be revealed in supersonic background flow, although the sound wave propagation is characteristically precluded there as well. For example, in a two-dimensional *unsteady* supersonic background flow an additional family of characteristics (along with the hyperbolic ones $\mathrm{d}x_j/\mathrm{d}t = u_j$) arises due to spatially hyperbolic properties like those in a *steady* supersonic flow [62].

At the second stage the variables $\mathbf{Z}_v(\mathbf{r}, t)$ are taken as known functions while one obtains the variables $\mathbf{Z}_\alpha(\mathbf{r}, t)$. The $\mathbf{Z}_\alpha$-system shows hyperbolic properties which correspond to both sound waves and some convection effects. Then the general sound source $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, q_s\}$, which arises in this system, can be correctly defined since it is determined solely by variables $\mathbf{Z}_v(\mathbf{r}, t)$. Note that the traditional concept of sound–flow interactions in inviscid gas media should be radically revised after such a decomposition. By the way, in section 4.4 the simplest model was presented in which a similar idea was used. While designing these two systems, which are normally of first order as the basic system (1)–(4), one must meet a substantial number of specific requirements in order to avoid possible ambiguities in such a decomposition: all the newly defined sources of aerodynamic sound must be integrable square over the infinite flow domain, the formulae for such sources are to be Galilean invariant, the norms of these sources should be minimized in G with the use of special *non-local* procedures to eliminate the spurious pseudosound effects, the boundary conditions have to be properly split as well, the relevant initial-boundary-value problems must conform to some important particular cases, and so on.

None of the existing theoretical models discussed subsequently in sections 5–7 has anything in common with this method. Anticipating further criticism, the author dares to say that those models, being unable to provide an adequate separation of sound field and unsteady background flow, cannot yield a correct definition of aerodynamic sound sources $\mathbf{Y}_s$ caused by both high-unsteady flow structure and the externally assigned source terms.

5. THE EXTERNALLY ASSIGNED SOURCES IN LINEAR MODELS

5.1. ON THE TRADITIONAL CONCEPT OF EXTERNAL SOURCES

The absence of a detailed analysis of diverse source terms is a serious limitation of Blokhintsev's linear model [2] in which the author has excluded this separate difficult problem. When unsteady body forces as well as mass and heat sources are assigned, one ought to give an exact procedure, and this is not trivial, for how to decompose each source term within a general procedure of flow decomposition (see equations (17) and (18)). This extremely important question is avoided even in the current publications on aeroacoustics, and its intricacy may be the reason for this.

This problem was considered by Goldstein in his well-known textbook *Aeroacoustics* [72] although his approach by no means can be accepted. In section 1.2 of reference [72] the assertion can be found that within the extended Blokhintsev model one should consider only those forces $\mathbf{F}$ and mass sources $\mu = \xi\rho$ (there $q = 0$ was assumed) which have "small" dimensionless values,

$$|\mathbf{F}(\rho_0 U_0 \omega_0)^{-1}| \leqslant O(\varepsilon), \quad |\mu(\rho_0\omega_0)^{-1}| \leqslant O(\varepsilon), \; \varepsilon = \|p_\varepsilon/p_0\| < 1,$$

where ω_0 is the characteristic frequency of disturbances. But what should one do if these values are not small? Generally, it seems unlawful to relate the source norms to the dimensionless amplitude of pressure disturbances ε, since the externally assigned source terms may be independent of the local flow character. Moreover, if their values are substantial, no linearized system can be used. At the same time, it was assumed that the mean flow variable $\mathbf{Z}_0(\mathbf{r})$ represented the steady solution of system (1)–(4) without any mass sources and forces

$$\rho_0(\mathbf{u}_0, \nabla)\mathbf{u}_0 + \nabla p_0 = 0, \tag{31}$$

$$\nabla(\rho_0 \mathbf{u}_0) = 0, \tag{32}$$

$$\mathbf{u}_0 \nabla s_0 = 0, \tag{33}$$

$$\Im(s_0, p_0, \rho_0) = 0, \tag{34}$$

Thereby, $\mathbf{F}$ and μ have been completely excluded from the mean-flow equations, being retained only in the linear equations which describe the evolution of small disturbances after decomposition (20), and just the latter were regarded as the *linear acoustic equations*.

For the simplest case of uniform mean flow with $\mathbf{u}_0 = \{U, 0, 0\}$ the following equation has been proposed in reference [72] (see equation (1.18)) to describe the sound field:

$$\Delta p_\varepsilon - \frac{1}{a_0^2}\frac{\mathrm{D}^2 p_\varepsilon}{\mathrm{D}t^2} = \nabla\mathbf{F} - \rho_0\frac{\mathrm{D}\xi}{\mathrm{D}t}, \quad \frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}. \tag{35}$$

Now one can easily understand why such an approach has been used. It is clear that time-dependent forces or mass sources act on a certain *unsteady background flow*, and at the same time they may generate sound waves, but no idea has been found how to define that unsteady background flow, much less within the non-linear system (1)–(4). So, if one retains any unsteady source term in the background-flow system, it is necessary to introduce the notion of unsteady background flow. Therefore, *all* unsteady effects of the source terms were considered there only within a linear system as small disturbances of steady flow.

By the way, this approach contradicts the classical concept of time averaging. Indeed, generally $\langle \mathbf{F} \rangle_t \neq 0$, and so the Blokhintsev model, where the gravity force was retained only in the mean-flow momentum equation, seems more progressive, at least in this respect.

As the simplest example which would disprove this approach, consider a flow where the gravitational mass force $\mathbf{g}$ is imposed (for instance, in the atmosphere). Then one comes to the paradoxical conclusion that mean flow is not affected by this force which can produce only "acoustic" effects. In this manner all convection phenomena caused by gravity should be attributed to acoustics!

Furthermore, this approach has resulted in the delusion that no sound waves are generated by force $\mathbf{F}$ if $\nabla\mathbf{F} = 0$. But a clear contrary example has been given in references [67, 88] where strong acoustic radiation occurs when the rotation in a round vortex is retarded by the external solenoidal force directed against the vector of rotational velocity. Besides, the *stationary* force with $\nabla\mathbf{F} \neq 0$ (as well as the mass sources when $\partial\xi/\partial t = 0$ but $U\partial\xi/\partial x \neq 0$ in equation (35)) may give a non-zero contribution to the "sound source term", and this fact does not enrich the approach.

Also, while considering the mass sources uniformly distributed in a finite domain ($\mu = \mu(t)$ in G), one would attribute them to the background flow rather than to the acoustic field.

The above approach, accepted and applied by many, gives no idea of how to separate the different actions of external source terms. The fact should also be taken into account that the classical model of incompressible fluid flow, which is often used for the approximation of steady subsonic mean flows does not imply the consideration of continuously distributed sources of both mass and entropy (although the new model of *globally compressible fluid flow* [86] enables one to do this).

Probably, this approach in flow acoustics originates from the conventional analysis of external sources in the linear acoustics of non-moving media (see, e.g., references [89–91]). Indeed, there all unsteady sources and forces were attributed to the acoustic field. Again this may be explained by the inability to introduce an accurate definition of unsteady background flow, even if rather slow, which can arise along with acoustic waves in the initially quiescent medium.

Following that classical way, one can write the linear system which describes the evolution of small disturbances $\mathbf{Z}(\mathbf{r}, t) = \{\mathbf{u}, s, p, \rho\}$ in a quiescent homogeneous background medium ($\mathbf{Z}_0 = \{s_0, p_0, \rho_0\}$) where *small* source terms $\mathbf{F}(\mathbf{r}, t)$, $q(\mathbf{r}, t)$, $\mu(\mathbf{r}, t)$ are the externally assigned functions in the domain G (clearly, one cannot linearize the basic system if those are not small):

$$\rho_0\, \partial\mathbf{u}/\partial t + \nabla p = \mathbf{F}, \quad \partial\rho/\partial t + \rho_0 \nabla\mathbf{u} = \mu, \tag{36, 37}$$

$$\partial s/\partial t = q, \tag{38}$$

$$p = (\rho_0 a_0^2/c_p)s + a_0^2 \rho, \quad a_0^2 = \gamma p_0/\rho_0 = \text{const.} \tag{39}$$

The closed system for $\mathbf{u}$ and p can also be written as

$$\rho_0\, \partial\mathbf{u}/\partial t + \nabla p = \mathbf{F}, \tag{40}$$

$$(1/a_0^2)\, \partial p/\partial t + \rho_0 \nabla\mathbf{u} = \zeta, \quad \text{where } \zeta = \mu + q\rho_0/c_p \tag{41}$$

Only if curl $\mathbf{F} = 0$, can one introduce the potential φ so that $\mathbf{u} = \nabla\varphi$, but in the general case the flow field may have non-zero vorticity. So the variable $\mathbf{Z} = \{\mathbf{u}, p\}$ is here influenced by the source $\mathbf{Y} = \{\mathbf{F}, \zeta\}$ while both sources μ and q act together through the value of ζ. From this system the following second-order linear equations can be derived

$$\Delta p - \frac{1}{a_0^2}\frac{\partial^2 p}{\partial t^2} = \vartheta = \nabla\mathbf{F} - \frac{\partial\zeta}{\partial t}, \quad \Delta = \operatorname{div}\operatorname{grad}, \tag{42}$$

$$\nabla^2\mathbf{u} - \frac{1}{a_0^2}\frac{\partial^2\mathbf{u}}{\partial t^2} = \mathbf{R} = \frac{1}{\rho_0}\,\nabla\zeta - \frac{1}{\rho_0 a_0^2}\frac{\partial\mathbf{F}}{\partial t}, \quad \nabla^2 = \operatorname{grad}\operatorname{div} \tag{43}$$

These equations, responsible for the evolution of field $\mathbf{Z} = \{\mathbf{u}, p\}$, describe diverse kinds of fluctuations, and not only sound waves, but this fact is often ignored.

What is most important is that from equation (42) the illusion may arise (as in equation (35)) that this equation is quite sufficient, and the only scalar source term ϑ may take the place of all the source functions $\mathbf{F}$, μ, q. However, generally one should specify each component from the complete set $\mathbf{Y} = \{F_1, F_2, F_3, \zeta\}$ in order to solve a relevant initial-boundary-value problem for $\mathbf{Z} = \{u_1, u_2, u_3, p\}$ within the closed system (40),(41).

Also, one should be very careful in treating the D'Alembertian on the left-hand side of equation (42) as "the sound propagation operator", because the properly assigned external sources are able to change radically all the characters of acoustic processes. So, in most general cases it would be better to regard the D'Alembertian merely as a certain set of spatial and temporal derivatives, since the formal availability of this operator does not imply automatically the presence of sound waves (this will be illustrated below). This concept will be followed further when the more complex non-linear equations with sound sources are considered.

Moreover, complex source functions are able to change the type of partial differential equations. This rather unusual assertion should be explained now. While making the local characteristic analysis of a certain system of partial differential equations (suppose the system to be of first order in both time and space like system (1)–(4)), one usually regards the source terms on the right-hand sides as the independently assigned *zero-order forcing terms* which can be omitted in the course of the analysis [92], although sometimes this may lead to unpredictable results. Indeed, the source terms, being complex functions of $\mathbf{r}$, t, Z_j, may also depend, perhaps "by chance", on $\partial Z_j/\partial t$ or $\partial Z_j/\partial x_i$, and then such terms, being of *first order* in fact, cannot be excluded from the analysis, because they are able to change radically the characteristic properties of the whole system. Just to avoid this ambiguity, it is often assumed that any source term depends only on $\{\mathbf{r}, t, Z_j\}$, but not on the temporal or spatial derivatives of Z_j. But conditions are quite possible when an externally assigned source function $Q(\mathbf{Z}, \mathbf{r}, t)$ may involve a term which can be treated as a function of $\partial Z_j/\partial t$ or $\partial Z_j/\partial x_i$.

As a simple example, one may consider the source function which is assigned in equation (42) as $\zeta = \zeta(p, \mathbf{r}, t)$. So the value of this function depends on the variable p, but the value of p *a priori* is unknown, since it can be found only from the solution of the whole initial-boundary-value problem. Nevertheless, suppose that the variable p depends on $\mathbf{r}$ and t as follows:

$$p = g(\mathbf{r})\varphi(t), \quad t \in J_p, \ \mathbf{r} \in G_p, \ G_p \subset G, \ J_p \subset J_t,$$

and then

$$\frac{\partial p}{\partial t} = g\frac{\mathrm{d}\varphi}{\mathrm{d}t} = p\mu, \quad \text{where } \mu = \mu(t) = \frac{1}{\varphi}\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \frac{\mathrm{d}\ln\varphi}{\mathrm{d}t}.$$

At the same time one could write $\zeta(p, \mathbf{r}, t) = \zeta(p\mu, \mathbf{r}, t)$, but this would mean that ζ depends on $\partial p/\partial t$, $\mathbf{r}$, t, and so the source function, now being of the first order in fact, is able to change the local characteristic properties of system (40),(41).

Thus, the volume sources open surprising opportunities in flow control. It should be noted that in section 3 of reference [67] the complete set $\{\mathbf{F}, \mu, q\}$ was regarded as a powerful control device which could allow one to obtain theoretically any desired evolution of all the flow variables within any basic system of differential equations, either linear or non-linear, and even irrespective of the type of that system. To show this, one may choose an arbitrary solution $\mathbf{Z}(\mathbf{r}, t) = \{\mathbf{u}, p, \rho\}$ in $G \times J_t$, and then solve the inverse problem by selecting the appropriate set of source terms $\{\mathbf{F}, \mu, q\}$ within system (1)–(4). Doing this, one formally allows those source functions to depend not only on $\mathbf{Z}$, $\mathbf{r}$, t, but on both temporal and spatial derivatives of $\mathbf{Z}$ as well, and then the characteristic type of our system may be changed radically. Unfortunately, in real flows one is usually unable, at least nowadays, to assign all the source terms within the whole domain G in an arbitrary manner.

5.2. ONE-DIMENSIONAL EXAMPLES

Consider the simplest one-dimensional problem for system (40), (41), posed in the finite interval $x \in (0, \pi)$, where, by specifying

$$F(x, t) = B[t^2 + 2a_0^{-2}] \sin x, \quad q = 0, \quad \mu = 0, \quad s = 0, \quad B = \text{const},$$

$$u(0, t) = u(\pi, t) = 0, \quad p(x, 0) = 0, \quad u(x, 0) = 0,$$

one obtains the following unique solution $\mathbf{Z}_\beta(x, t)$;

$$p_\beta(x, t) = -Bt^2 \cos x, \quad u_\beta(x, t) = 2Bta_0^{-2}\rho_0^{-1} \sin x, \quad t \in (0, t_f),$$

which by no means resembles the acoustic oscillations in a closed volume. Surely, both B and t_f should be taken in such a manner that the condition $|B|t_f^2 a_0^{-2} \rho_0^{-1} \ll 1$ is satisfied (otherwise the problem may become non-linear).

If one assigns $p(x, 0) = A \cos x$, a standing sound wave will additionally arise in the domain,

$$p_\alpha(x, t) = A(\cos a_0 t) \cos x, \quad u_\alpha(x, t) = A a_0^{-1} \rho_0^{-1} (\sin a_0 t) \sin x,$$

while the variables $\mathbf{Z}_\alpha(\mathbf{r}, t) = \{u_\alpha, p_\alpha\}$ are described by the system of homogeneous equations

$$\rho_0 \frac{\partial u_a}{\partial t} + \frac{\partial p_a}{\partial x} = 0, \quad \frac{1}{a_0^2}\frac{\partial p_a}{\partial t} + \rho_0 \frac{\partial u_a}{\partial x} = 0,$$

supplemented by conditions $u_\alpha(0, t) = u_\alpha(\pi, t) = 0$, $u_\alpha(x, 0) = 0$. Hence, one can denote the total solution as $\mathbf{Z}^*(\mathbf{r}, t) = \mathbf{Z}_0 + \mathbf{Z}_\beta + \mathbf{Z}_\alpha$. Then the *non-acoustic* part $\mathbf{Z}_0 + \mathbf{Z}_\beta$ may be treated as *unsteady background flow* for the sound oscillations $\mathbf{Z}_\alpha$. What is also important is that by assigning different values of A and B, one can vary the ratio A/B in a wide range (for instance, $|A/B| \ll 1$ could be taken). Of course, only within the linear model can the sound phenomena be considered separately without any influence of force F. Probably, if one applies a more general non-linear model, then an unsteady background flow like $\mathbf{Z}_0 + \mathbf{Z}_\beta$ is able to produce additional sound sources, which in turn will change the acoustic field $\mathbf{Z}_\alpha$.

In comparison, one can consider another solution $\mathbf{Z}^*(\mathbf{r}, t)$ within $x \in (0, \pi)$, $t \in (0, t_f)$:

$$F = 0, \quad \zeta = -B[2ta_0^{-2} + t^3/3]\cos x,$$

$$p(x, 0) = A\cos x, \quad u(x, 0) = 0,$$

$$p_\beta(x, t) = -Bt^2\cos x, \quad u_\beta(x, t) = -Bt^3(3\rho_0)^{-1}\sin x,$$

$$p_\alpha(x, t) = A(\cos a_0 t)\cos x, \quad u_\alpha(x, t) = Aa_0^{-1}\rho_0^{-1}(\sin a_0 t)\sin x.$$

Here again the source term $\zeta(\mathbf{r}, t)$ influences only the non-acoustic components $\mathbf{Z}_0 + \mathbf{Z}_\beta$. Moreover, the rather delicate question may be posed whether the well-known Rayleigh criterion is applicable to this case. At least the area of its applicability should be defined more accurately. Indeed, if one analyzes solely the evolution of the acoustic component $\mathbf{Z}_\alpha(\mathbf{r}, t)$, then the mass and/or heat source term ζ does not cause any changes in $\mathbf{Z}_\alpha$, although for $T = 2\pi/a_0 < t_f$ one has

$$J_s = \int_0^\pi \int_0^T p_\alpha \zeta \, \mathrm{d}t \, \mathrm{d}x = -\frac{2\pi^3 AB}{a_0^4} > 0 \quad \text{if } -AB > 0.$$

Spherically symmetric solutions can be found in an infinite spatial domain where the effects of sound radiation are completely eliminated by assigning properly the source terms in the finite volume $G_s = \{r < \pi/2\}$. In this particular case the linear system (40),(41) reduces to

$$\rho_0 \frac{\partial u}{\partial t} + \frac{\partial p}{\partial r} = F, \quad \frac{1}{a_0^2}\frac{\partial p}{\partial t} + \frac{\rho_0}{r^2}\frac{\partial r^2 u}{\partial r} = \zeta.$$

For instance, the exact solution

$$p = A(\cos\omega t)\cos^4 r, \quad u = 4A\omega^{-1}\rho_0^{-1}(\sin\omega t)\sin r\cos^3 r, \quad \text{when } r < \pi/2,\ t > 0,$$

$$u = p = 0 \quad \text{when } r > \pi/2,\ t > 0,$$

is attained if one assigns

$$F = 0,\ \zeta = 4A\omega^{-1}(\sin\omega t)\ [2r^{-1}\cos^3 r\sin r + \cos^4 r - 3\cos^2 r\sin r]$$

$$- A\omega a_0^{-2}(\sin\omega t)\cos^4 r, \quad \text{when } r < \pi/2, \tag{44}$$

$$F = \zeta = 0 \quad \text{when } r > \pi/2.$$

In contrast to the previous example, one now can consider the oscillating source ζ which may be implicitly related to $\partial u/\partial t$, $\partial u/\partial r$, $\partial p/\partial t, \ldots$. The initial conditions are then specified as

$$p(r, 0) = A\cos^4 r,\ u(r, 0) = 0, \quad \text{when } r < \pi/2,$$

$$p(r, 0) = 0, \quad u(x, 0) = 0, \quad \text{when } r > \pi/2.$$

Expressions (44) can be simplified if one takes $\omega = 2a_0$. This gives

$$\zeta = 4A\omega^{-1}(\sin\omega t)[2r^{-1}\cos r - 3]\cos^2 r\sin r \quad \text{when } r < \pi/2,\ t > 0.$$

Thus, in this exotic example no sound waves are emitted from the region $r < \pi/2$ where the oscillating mass and/or heat sources occur.

Important general conclusions can be drawn from the above linear examples. The presence of mass and heat sources as well as external forces has to change one's habitual comprehension of acoustic processes, especially if one analyzes these by following the most accurate way, i.e., by obtaining the unique solution of quite general initial-boundary-value problem. The particular source functions are even able to change the characteristic properties of the basic system of partial differential equations. When one simulates the evolution of complex initial disturbances, perhaps with non-zero vorticity, the presence of the D'Alembertian on the right-hand side of equation (42) does not guarantee the usual scenario of sound propagation (i.e., the source terms are able to suppress all sound waves, or to change both the velocity and direction of their propagation, etc.). Hence, a number of unambiguous requirements for the source functions $\{\mathbf{F}, \mu, q\}$ should be formulated (for instance, one has to assume zero-order relations of these latter to the main flow variables within the basic system (1)–(4)) in order to restrict the *unlimited control abilities* of source terms.

So the source terms may influence *non-acoustic* disturbances rather than acoustic waves (surely this depends on the form of source function in part), and this fact contradicts the conventional approach [72] where all external source terms act only on the sound field. These examples emphasize again the key problem of accurately defining both the unsteady background flow and the acoustic field.

By the way, pronounced imperfection can be found in the existing theoretical models which are applied to the simulation of sound effects due to thermal sources. A long-standing opinion exists that most thermal phenomena, including strong heat conduction, volume heat release, entropy production due to mass sources, etc., can be simulated only within the general model of compressible fluid flow. However, the model of globally compressible fluid flow [86], that represents a fundamental extension of the classical model of incompressible fluid flow, enables one to simulate the main thermal phenomena in subsonic flows by operating with all the usual set of thermodynamic relations, while the sound effects are characteristically excluded. Thereby, it has been demonstrated that many thermal processes can be well simulated without any connection with sound.

Consequently, the universally accepted models for the sound generation due to externally assigned sources and forces cannot be recognized as satisfactory.

An additional number of sharp questions will arise if one tries to specify the appropriate set of boundary conditions on permeable or moving surfaces within a certain aeroacoustic theory which is based on flow decomposition. Then one should distinguish the actions of boundary conditions on the unsteady background flow and on the acoustic field. One can try to do this only after giving an accurate definition of unsteady background flow in all the spatial domain within that model, but even then quite different ways may exist for the decomposition of boundary values of the flow variables into the background-flow components and the acoustic and other disturbances within the same system of governing equations. The routine procedure of time averaging, like that applied to the basic equations at internal points, may be used on the boundary as well, but it is hardly possible to find the best version of that procedure because all its details may be decisive. Many examples can be given where "small" changes in the boundary conditions are able to cause intense acoustic resonances. All these difficulties explain why the effects of unsteady boundary conditions in aeroacoustics are yet often studied within a linear model where the evolution of small fluctuations, which include all kinds of waves, is simulated on the background of steady mean flow.

6. LIGHTHILL'S ACOUSTIC ANALOGY

Reference [45], published in 1952, is the first and the most influential attempt to create a general theoretical model for sound generation by flow. Let us discuss again that approach applied to our case of inviscid gas flow. To do this, one can follow in all details the classical way described in many textbooks (e.g., in reference [72]) of deriving this famous second-order equation; note first, however, that only equations (1),(2) are used for this, and equations (3),(4) are not included in the derivation process.

To begin with, one obtains a new version of the momentum equation after adding the term $a_0^2 \nabla \rho_\varepsilon$ to both sides of equation (1). Then the operator $\partial/\partial t$ is applied to equation (2), and the operator ∇ to the new version of equation (1). Taking the difference of these second-order equations yields

$$\frac{\partial^2 \rho_\varepsilon}{\partial t^2} - a_0^2 \Delta \rho_\varepsilon = Q_L = \frac{\partial^2 T_{ij}}{\partial x_i x_j} + \frac{\partial \mu}{\partial t} - \frac{\partial \mu u_i}{\partial x_i} - \frac{\partial F_i}{\partial x_i}, \qquad (45)$$

where $\rho_\varepsilon = \rho - \rho_0$, $p_\varepsilon = p - p_0$, and $T_{ij} = \rho u_i u_j + \delta_{ij}[p_\varepsilon - a_0^2 \rho_\varepsilon]$ is *Lighthill's stress tensor*. As a result, all the non-linear right-hand part Q_L was called in reference [45] as a "quadrupole sound source". In another manner one can write

$$Q_L = \nabla[\nabla(\rho \mathbf{u}; \mathbf{u}) + \nabla(p_\varepsilon - a_0^2 \rho_\varepsilon)] + \partial \mu/\partial t - \nabla(\mu \mathbf{u}) - \nabla \mathbf{F}.$$

Note that almost everywhere this equation has been mistakenly written without the term $\nabla(\mu \mathbf{u})$ (see, e.g., equation (2.5) in reference [93]), and this may be explained by incorrect specification of the vector $\mathbf{k}$ in equation (1).

There is no reason for regarding ρ_ε and p_ε as the "acoustic components" because no accurate definition of acoustic components in a high-unsteady flow has been given in reference [45], so that ρ_ε and p_ε represent merely the differences between the local values of flow variables $\{\rho, p\}$ and *arbitrary* constants $\{\rho_0, p_0\}$. It is not necessary to give a physical definition of the constants $\{p_0, \rho_0, a_0\}$ in order to motivate the above mathematical transformations; all the same, equation (45) will remain valid at any values of these constants one specifies. Note that the values of p_0 and ρ_0 are absolutely unimportant since both new variables ρ_ε, p_ε are under differentiation operators. But the constant a_0 does attract more attention. Indeed, the crucial, and the most feeble, point of this approach is that *after* deriving equation (45) the left- and right-hand part of this equation are considered *separately* (!), and then the constant a_0 is able to play decisive roles in them. If one regards equation (45) at $Q_L = 0$ as something like the "classical wave equation for the density disturbances in a quiescent homogeneous medium", the reasons for a choice of a_0 are most unclear. Formally, one has a right to specify $a_0 = 0$, or $a_0 = 1$, or something else. For instance, constants $\{p_0, \rho_0, a_0\}$ may be defined *after* obtaining equation (45) as the pressure, density, and the velocity of sound in a quiescent background medium far from the confined region G_f with high-unsteady flow, as was suggested in reference [45]. In any case, the constant a_0 does not represent the local value of the speed of sound, much less when substantial gradients of temperature are present. If someone insists that these constants $\{p_0, \rho_0, a_0\}$ must be taken in a unique manner as a set of thermodynamic parameters at a *definite* point outside G_f, it is extremely doubtful. Probably, one could find another remote point in a spot of quiescent hot gas where $p_1 = p_0$, $\rho_1 = \rho_0/4$, $a_1 = 2a_0$. However, a new constant a_1 will change radically the left-hand part of equation (45). It seems that these constants may be helpful only as the characteristic values which one can use in composing a set of dimensionless variables instead of dimensional ones (then the local values of flow velocity as well as the local speed of sound could be related to a_0).

So the non-linear expression Q_L has been defined in reference [45] as a "sound source", although it is impossible to explain logically why the right-hand part of equation (45) is taken as a source for the left-hand part, when both these are second-order differential expressions of equal standing. Thereby, this interpretation of Q_L contradicts the classical notion of a source as an externally assigned function (see section 5). Generally, there is no point in considering this equation separately since it includes the complete set of *unknown total variables* $\mathbf{u}$, p, ρ, and one cannot determine any variable in this set before obtaining an exact solution within the closed system (1)–(4). Besides, there are no grounds to treat equation (45) as a non-homogeneous equation of hyperbolic type in time and space. One can analyze the characteristic properties of a closed system, but not a separate scalar equation with several variables, and below an example will be given where this equation remains unchanged, but the system displays quite different characteristic features.

But if all the definite initial-boundary-value problem has been completely solved with the use of closed system (1)–(4), the need for the additional equation (45) is questionable, since that by no means helps one to separate out the acoustic disturbances from an unsteady background flow. Presumably, if one succeeds in deriving a self-closed high-order equation for a single scalar variable (though this seems to be highly improbable), even then it will be impossible to analyze acoustic phenomena without separating out the sound component from that variable.

One of the most dramatic errors should also be noted: it is absolutely insufficient to define an expression like Q_L as *the only source term* in a sole second-order scalar equation. The formulae to be found must include all members of the complete set of sound sources $\{\mathbf{F}_s, \mu_s, q_s\}$ while non-linear acoustic equations form a closed first-order system (see equations (18), (36)–(43)), and if such a system has been created, then one may do without deriving any subsequent second-order equations.

In vain efforts to make the model more plausible, it has been suggested to consider a *compact* region G_f (with size $l \ll \lambda$ where λ is the wave length of the sound) which contains non-uniform unsteady flow, and this region is surrounded by an infinitely extended homogeneous gas medium with zero mean velocity and constant values a_0, p_0, ρ_0. Although such an ideal flow configuration is far from the reality of most flows under study, clearly outside region G_f, equation (45) with $Q_L = 0$ can be used to describe small acoustic disturbances in a quiescent gas medium. But this fact is absolutely unable to prove that just in G_f all the right-hand side of equation (45) can be interpreted as a source term responsible for the generation of sound by flow. Generally, it seems senseless to demand the compactness of a certain flow domain (as usual that is unrealizable within all possible values of λ, much less if one does not know in advance the spectrum of generated sound) as a necessary condition of the model's validity if this model aspires to be quite general. This condition may be useful only when one tries to calculate the averaged strength of sound sources in G_f *after* having obtained the distribution of those sources all over G_f. Irrespective of the domain size, one should produce an accurate method of determining the sound sources at each point of G_f. Thus, it is no use mentioning further a "quiescent external medium" which by no means is related to the local processes of sound generation.

If one considers possible effects of external forces and mass sources in equation (45), a curious peculiarity will arise: the external source terms μ and $\mathbf{F}$ contribute to the sound field only through the joint scalar term $\vartheta = \partial\mu/\partial t - \nabla(\mu\mathbf{u}) - \nabla\mathbf{F}$. As in the linear model of reference [72] one may come to the wrong conclusion that force $\mathbf{F}$ does not cause any sound sources when $\nabla\mathbf{F} = 0$ (one can recall again the contrary example given in reference [88] where strong sound radiation occurs due to a solenoidal force imposed on a round vortex). Thus, only the scalar term $\nabla\mathbf{F}$ seems to be decisive within this approach, but the different values of components $\{F_1, F_2, F_3\}$ seem unimportant. Besides, if a steady flow is investigated where $\partial\mu/\partial t = 0$ and $\partial\mathbf{F}/\partial t = 0$, the term $-\nabla(\mu\mathbf{u}) - \nabla\mathbf{F}$ is able to give a non-zero contribution to Q_L, but this contradicts the universal opinion that no sound sources are present in such a flow. A striking fact should be distinguished as well: equation (3) has not been used in deriving equation (45), and so the entropy source q, in contrast to μ and $\mathbf{F}$, is unable to generate sound. This however does not conform to our experience in the study of thermal sources, even within the linear acoustics of non-moving media (see section 5). One can say that all effects of the external sources are included in the exact solution $\mathbf{Z}(\mathbf{r}, t) = \{\mathbf{u}, s, p, \rho\}$ to a certain initial-boundary-value problem posed for the closed system (1)–(4). But if such a solution is known, one can do without the *dependent* equation (45) since the latter does not provide any helpful additional data on the aeroacoustic phenomena.

Another essential flaw must be mentioned: expression Q_L, as well as the left-hand side of equation (45), are not Galilean invariant, but it would be much better if the values of aerodynamic sound sources were independent of the reference frame in the same manner as the external source terms $\mathbf{F}$, μ and q (at least, it would be very helpful in comparing the effects of "external" and "internal" sources). Indeed, the components of a hypothetical expression for the aerodynamic sound source $\mathbf{Y}_s = \{\mathbf{F}_{s1}, \mathbf{F}_{s2}, \mathbf{F}_{s3}, \mu_s, q_s\}$ act as mass forces, mass production and entropy sources, and these may be treated as some additives to the externally assigned terms $\{\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3, \mu, q\}$. Clearly, this flaw has arisen because the non-invariant operator $\partial/\partial t$ was applied to equation (2). Such a procedure, used for obtaining the well-known expression on the left-hand side of equation (45), seems to be inherited directly from the classical acoustics of quiescent media, but this way seems to be unpromising in aeroacoustics. As a result, the value of the term $\partial\mu/\partial t - \nabla(\mu\mathbf{u})$, that is assumed to assess the contribution of mass sources to the sound generation, changes according to the choice of the reference frame. By the way, that notion of a quiescent gas medium, which surrounds the "compact flow region with sound sources", might be introduced with the additional aim of defining the unique reference frame and thereby avoiding all questions on the non-invariance of both sides of equation (45). Nevertheless, if one considers a stable vortical structure convected by flow in that "compact region", this particular case does not result definitely in a zero value of Q_L.

Undoubtedly, all these defects put Lighthill's approach in a bad light. To complete the gloomy picture, one can undertake a curious experiment which reveals the most serious conceptual error of that "acoustic analogy". Let us take the non-local model of unsteady subsonic flow [86] where all sound effects are characteristically precluded (although the value $a^2 = \gamma p_c/\rho$ is finite there!). The total value of static pressure p_c is there decomposed as

$$p_c(\mathbf{r}, t) = P(t) + p(\mathbf{r}, t),\ P > 0,\ \max|p/P| = O(\varepsilon) \ll 1. \tag{46}$$

Then, instead of equations (1)–(4), one has the new system of governing equations

$$\partial\rho\mathbf{u}/\partial t + \nabla(\rho\mathbf{u};\mathbf{u}) + \nabla p = \mathbf{F} + \mathbf{k}, \tag{47}$$

$$\partial\rho/\partial t + \nabla(\rho\mathbf{u}) = \mu, \tag{48}$$

$$\partial s/\partial t + \mathbf{u}\nabla s = q, \tag{49}$$

$$\Im(s, P, \rho) = 0, \tag{50}$$

supplemented by the special algorithm for calculating $P(t)$ (see reference [86]).

One can apply a similar procedure to equations (47),(48) as that applied to equations (1),(2). Then a surprising result is obtained: one derives a second-order equation which is exactly equivalent to equation (45)! Again the linear expression $\Box\rho_\varepsilon$ forms the left-hand side of that equation where $\Box = \partial^2/\partial t^2 - a_0^2\Delta$ is similar in appearance to the classical sound propagation operator. However, no sound waves can exist in such a new model where the modified equation of state (50) is used along with a system like equations (1)–(3). Clearly, independent basic equations (1),(2) as well as equation (45) are not influenced by any subsequent changes in the equation of state. So addition of the term $a_0^2\nabla\rho_\varepsilon$ to both sides of equation (1), as well as the specification of any constant a_0, do not warrant the existence of sound waves. This paradoxical example looks quite sufficient to disprove the Lighthill concept of sound sources.

Generally, the model of reference [86] contributes greatly to aeroacoustics, since it shows that many phenomena in subsonic flows, which were mistakenly regarded to be connected with sound effects, may be well simulated with acoustics excluded. Besides, that model refutes the long-standing opinion that one can operate all the set of thermodynamic relations in a gaseous medium with $\gamma \neq 1$ only if the general model of compressible fluid flow is applied.

Furthermore, one can take the classical model of incompressible fluid flow without mass sources [94] as a particular case of the model of reference [86], and then the closed system for $\mathbf{Z} = \{\mathbf{u}, p, \rho\}$ can be written as

$$\partial\rho\mathbf{u}/\partial t + \nabla(\rho\mathbf{u};\, \mathbf{u}) + \nabla p = \mathbf{F}, \tag{51}$$

$$\partial\rho/\partial t + \nabla(\rho\mathbf{u}) = 0, \tag{52}$$

$$\nabla\mathbf{u} = 0, \tag{53}$$

where the thermodynamic equation of state is not used at all, and pressure p is determined to within an arbitrary additive function $p_a(t)$. If the same procedure is applied to equations (51) and (52), then one will obtain the celebrated equation (45) as well, although now without the term $\partial\mu/\partial t - \nabla(\mu\mathbf{u})$ on the right-hand side. Note again that this approach does not need any other equations except equations (1),(2), and constants $\{p_0, \rho_0, a_0\}$ can be specified in an arbitrary manner. So it has been shown that the "non-homogeneous acoustic equation" (45) can be derived within the model of incompressible fluid flow.

The paper by Crow [95] is often cited as the most comprehensive and rigorous substantiation of Lighthill's approach. After reviewing some other models of aerodynamic sound sources in the introduction, the author's conclusion was there expressed that "none of these alternative physical descriptions has been able to compete with Lighthill's theory". One can touch upon the key idea of this work, although any subsequent proofs are evidently unable to justify the basic "acoustic analogy" which is flawed by the ineradicable defects mentioned above. The infinite spatial domain G_∞, which contains the local region G_f with vortical homentropic subsonic flow, is considered in reference [95], and the total velocity is there split as equations (22). The solenoidal component $\mathbf{u}_s$ directly related to the vorticity $\boldsymbol{\omega}$ in G_f through the Biot–Savart law (it was assumed that at infinity the velocity decreases as $|\mathbf{r}|^{-3}$) was attributed to the background flow, and the irrotational part $\mathbf{u}_p = \nabla\varphi$ was regarded as the "acoustic component". Surely, decomposition (22) is applicable to any velocity field, and many other works could be mentioned where a similar procedure has been used, but the uniqueness of this procedure in definite flow conditions demands special analysis. Generally, $\mathbf{u}_p$ does not consist solely of the sound-wave component, and the latter may have non-zero vorticity. No accurate method was given how to decompose both density and pressure into acoustic and background-flow components within a general non-linear problem, so that p and ρ were merely attributed in reference [95] to the sound field (?). Note that in general non-linear formulation of this approach the solenoidal velocity field was not independent, and its evolution was much influenced by the "acoustic" variables p and φ. Then the method of matched asymptotic expansions was applied in efforts to estimate the sound sources defined by equation (45), and in turn the far sound field, on the basis of the vorticity distribution in G_f. Since no independent closed system was proposed in reference [95] to obtain accurately $\{\boldsymbol{\omega}, \mathbf{u}_s\}$, the routine way was used: $\mathbf{u}_s$ had to be roughly approximated by using an appropriate solution to the related problem posed within the model of incompressible fluid flow.

Thus, Crow's analysis looks too approximate, so that it is easy to lose the small-amplitude sound disturbances in the long logical way to the final estimates, especially due to the number of questionable assumptions and assertions. So it seems to be of no use to discuss the conclusions of this work because the detailed critical analysis of the basic Lighthill approach has been made above. However, the following phrase in the concluding remarks of reference [95] should be cited as rather curious: "In any case, the problem of aerodynamic sound is not closed by the assertion that T_{ij} accounts for all the phenomena of compressible, rotational flow. T_{ij} is *sterile abstraction* insofar as those phenomena are not separately understood".

Nevertheless, one may try to guess the main reason why this model even nowadays continues to have many followers. Probably, the irresistible attractiveness of this approach is explained by the illusory simplicity of equation (45). Indeed, at first sight equation (45) may resemble the routine linear acoustic equation with *external source* Q_L, but only if one ignores all fundamentals of mathematics and mathematical physics. The obtaining of the linear expression $\Box\rho_\varepsilon$ on the left-hand side of a possible second-order equation seems to be the key idea of the approach, although "the classical operator of sound propagation" $\Box$ corresponds to the fictitious quiescent medium but not to a real complex unsteady flow in G_f. Evidently, this primitive idea is unable to yield an adequate theory of sound generation and propagation, because no accurate definition of acoustic components in unsteady background flow has been suggested. Moreover, as shown in the examples given in section 5.2 as well as when equation (45) was derived from system (51)–(53), the availability of the "sound propagation operator" on the left-hand side of a separately taken second-order "inhomogeneous wave equation" does not guarantee the presence of the usual mechanism of sound propagation. In this way, a linear part of any desired form can be separated out (which may contain the D'Alembertian in particular) from *any* equation if one adds something to both sides of that equation and then transfers all the rest of the non-linear differential expressions to the right-hand side in order to call them a "source". This approach also offers a false idea to retain *the only scalar variable* ρ_ε in the left-hand linear part, so that all other variables can be placed on the right-hand "source", and the latter is to be approximated with the use of some "external" data. So an illusion may arise that *any* aeroacoustic phenomenon can be described by a sole scalar variable! Then all the vast experience in the theory of linear partial differential equations suggests itself to the solution of such a converted problem (one may try to derive an appropriate Green function, and so on). The author thinks that any mathematician has to be greatly surprised that such a "powerful" method has not yet been widely applied to the solution of *any* non-linear differential equations.

Following the above concept, one may design a number of "alternative" models. For instance, equation (45) could be rewritten as

$$\Delta p_\varepsilon = Q_E, \quad Q_E = -\frac{\partial^2 T^0_{ij}}{\partial x_i x_j} + \frac{\partial^2 \rho_\varepsilon}{\partial t^2} - \frac{\partial \mu}{\partial t} + \frac{\partial \mu u_i}{\partial x_i} + \frac{\partial F_i}{\partial x_i}, \tag{54}$$

where $T^0_{ij} = \rho u_i u_j$. This equation, which is actually equivalent to the version given in reference [96], may be treated as an elliptic Poisson's equation with the "source" Q_E. What a pity: now the left-hand side does not contain the "sound propagation operator".

Equation (45) can be also written in another form

$$\partial\rho_\varepsilon/\partial t - a_0^2 \Delta\rho_\varepsilon = Q_P, \tag{55}$$

where

$$Q_P = \frac{\partial^2 T_{ij}}{\partial x_i x_j} - \frac{\partial^2 \rho_\varepsilon}{\partial t^2} + \frac{\partial \rho_\varepsilon}{\partial t} + \frac{\partial \mu}{\partial t} - \frac{\partial \mu u_i}{\partial x_i} - \frac{\partial F_i}{\partial x_i}.$$

Then one may be enticed to regard equation (55) as a parabolic equation with the "source term" Q_P. Now the ambiguous constant a_0^2 acts as a diffusion coefficient.

Note also that a kind of "non-homogeneous wave equation" was actually found by Blokhintsev in the 1940s, much before Lighthill's approach appeared. Indeed, equation (12) can be rewritten as

$$\frac{\mathrm{d}^2\varphi_\varepsilon}{\mathrm{d}t^2} - a_0^2 \Delta\varphi_\varepsilon = Q_\varphi, \tag{56}$$

and then someone is able to regard the expression

$$Q_\varphi = (\nabla h_0,\ \nabla\varphi_\varepsilon) + \frac{\mathrm{d}\varphi_\varepsilon}{\mathrm{d}t}(\mathbf{u}_0,\ \nabla\ \ln a_0^2)$$

as a sound source in an irrotational homentropic flow.

Thus, the family of "alternative" second order equations (54),(55), as well as the basic equation (45), all result from the same general concept, and so all those represent equally absurd approaches to the theory of aerodynamic sound sources.

So the true value of equation (45) for aeroacoustics is questionable, although innumerable followers have taken up this "revolutionary" idea without paying attention to the above defects (or most likely, not having detected all those defects), and their further efforts have been directed mainly to possible modifications of the "quite general" equation (45) and its adaptation for diverse particular cases. Unfortunately, a number of influential persons in aeroacoustics have expressed the highest appreciation of this model ("*Lighthill's theory has been by far the most successful and versatile*" [97], "*That was a masterpiece*" [16], "*The Lighthill theory of aerodynamic noise is the most important advance in acoustics since the work of Lord Rayleigh*" [39], …); such endorsements have misled a lot of young scientists, in the author's opinion.

7. ON SOME OTHER APPROACHES TO THE THEORY OF AERODYNAMIC SOUND

We could do without a review of numerous subsequent works which developed, modified and simplified the "quite general" approach [45]. In fact, all principal defects of the basic "acoustic analogy" have been inherited in these because nobody has really disputed the key concept of aerodynamic sound sources given by means of equation (45). Nevertheless, a few of these works, which are widely known amidst the major theoretical methods in aeroacoustics, will be mentioned below to fill a historical chronicle.

As a "generalization" of Lighthill's model to include aerodynamic surfaces in motion, a special scalar equation, which may be written either in differential or in integral form, has been proposed by Ffowcs-Williams and Hawkings [52] where three terms on the right-hand side are treated as the thickness (or monopole), loading (or dipole), and quadrupole-like source terms. This well-known equation is based completely on the "quite general" approach [45], and so its analysis would yield the same critical conclusions as those given in section 6. This equation is often used as the key tool for the prediction of noise produced by rotating propeller blades, including helicopter rotors and fans. In a number of further works [14, 20, 98–100] modified versions of that equation as well as the relevant computational codes were suggested, including those applied for sonic and supersonic surface motions in order to avoid some singularities. It should be noted that no consistent theoretical concept of sound generation has been offered specifically for transonic and supersonic flows, and so if approach [45] is applied to such flows, then new difficult questions arise in addition to those discussed above.

Most other works can be classified according to the modifications made in three main directions: (i) the basic expression Q_L is simplified and approximated, mainly for subsonic flows; (ii) other variables are used, and another exact scalar second-order equation is composed instead of equation (45) with the aim of finding "an adequate sound propagation operator" on the left-hand side; (iii) new non-linear or linearized equations with "sound sources" on the right-hand side are derived for small disturbances $\mathbf{Z}_\varepsilon$ by applying the time-averaging procedure (13) and decomposition (20).

Concerning the second way, the main mistake can be readily found in such efforts to "generalize" the Lighthill acoustic analogy. In equation (45) the right-hand side has been suggested to be regarded as the *only* scalar sound source Q_L on the ground that the *only* scalar variable ρ_ε is present on the left-hand side. Though, as shown above, the whole of this model is wrong, this fact, considered separately, may seem at least non-contradictory. But in any subsequent approach, when one tried to derive a modified second-order equation with "the true propagation operator", new independent variables ($\mathbf{u}$ and a as minimum) were usually brought into the left-hand side, and then it seems absolutely insufficient to specify a sole second order "wave equation" with a sole scalar "sound source". In any case, it should be particularly emphasized that in designing such a left-hand side of a "second order scalar inhomogeneous wave equation" so that it would contain something like the "sound propagation operator" (see sections 5 and 6), one is unable to provide any rigorous proofs for the definition of all right-hand side as a "sound source".

Here it is relevant to give a brief comment on both known and prospective attempts to design *other* systems of aeroacoustic equations. Of course, one may substitute the basic non-linear system of governing equations written for the usual set of total variables $\{\mathbf{u}, s, p, \rho\}$ (for instance, system (1)–(4)) by a new, perhaps much more complex, system of equations in terms of another set of total variables or for fluctuating parts of those variables. Further, one can try to compose something like a new "true" form of "inhomogeneous scalar acoustic equation", although the habitual method may be used: all *inconvenient* non-linear terms are transferred to the right-hand side of that equation and interpreted as a total sound source, and so on. However, one is able to continue such an activity without any progress in the long-standing question of great importance: in what manner is it possible to derive a *closed* system of non-linear equations within which all components of sound disturbances are separated out accurately in a high-unsteady flow?

7.1. MODELS OF SOUND GENERATION BY LOW MACH NUMBER FLOWS

According to reference [45], the important particular case of homentropic subsonic flow (at Mach number $M = \max|\mathbf{u}/a|_G \ll 1$) can be considered. Then p_ε and ρ_ε are defined as the small disturbances in uniform mean flow with constant values p_0 and ρ_0, so that one can write

$$p_\varepsilon = p - p_0 \approx a_0^2(\rho - \rho_0).$$

Note that now the constant $a_0^2 = \gamma p_0/\rho_0$, in contrast to that in equation (45), is quite definite. For this case in references [45, 46], equation (45) without any external sources and forces was suggested to be approximated as

$$\Box \rho_\varepsilon \approx Q_\beta, \quad Q_\beta = \rho_0 \nabla[(\mathbf{u}_\beta, \nabla)\mathbf{u}_\beta] = -\Delta p_\beta, \quad \nabla \mathbf{u}_\beta = 0. \tag{57}$$

So, to estimate the "unified scalar sound source" Q_L while solving a certain problem within the model of a compressible medium, one should determine the non-linear expression Q_β from the previously found solution $\mathbf{Z}_\beta = \{\mathbf{u}_\beta, p_\beta, \rho_0\}$ of the *related* (?) evolutionary problem posed within the quite different model of incompressible fluid flow (with different characteristic properties), but possible relations, both qualitative and quantitative, between those problems are too ambiguous, and this key question has not been discussed in detail.

The weakest point of this approximation must be pointed out, in addition to the principal defects of the basic Lighthill model. One separates out the "main part" of Q_L by omitting the "small difference" ε between Q_L and Q_β. Thereby, the acoustic components are regarded as absolutely negligible on the right-hand part, but it would be quite logical to do the same within the left-hand side of equation (45) as well. There it was assumed that $|\varepsilon| \ll |Q_L|$, but if one intends to use equation (57) for the analysis of sound processes, $|\varepsilon|$ may not be small in comparison with some terms containing the acoustic variables, and so ε cannot be neglected in the whole equation. If the norms of acoustic disturbances are much less than the norms of mean-flow variables, and if one neglects those former *everywhere*, this procedure seems more appropriate for the approximation of unsteady background flow but not that of the sound field. Such a means of sound filtering in both sides of equation (45) is able to yield a non-zero value of Q_β, but the latter will have no connection with sound sources. Here one can note the example given in the previous section where equation (45) was derived from system (51)–(53) governing incompressible fluid flow. Meanwhile, a curious "half-advance" has been attained in this approach: the right-hand side of equation (57) has acquired the Galilean invariant form although the left-hand side remains non-invariant.

In developing this idea, Powell [49] has proposed a further "simplification" of Lighthill's approach. Expression (57) for Q_β was rewritten as

$$Q_\beta = \rho_0 \nabla[(\nabla \times \mathbf{u}_\beta) \times \mathbf{u}_\beta + \nabla \mathbf{u}_\beta^2/2],$$

Then it was asserted that in an unbounded domain the second term on the right-hand side produced a far sound field which was much less intense than that associated with the first term, and so the second term was suggested to be omitted. As a result, the following approximation was suggested:

$$Q_\beta \approx \rho_0 \nabla[(\nabla \times \mathbf{u}_\beta) \times \mathbf{u}_\beta] \tag{58}$$

However, the coarse estimates, made in section 1.3 of reference [49] on the basis of characteristic Mach and Strouhal numbers, are insufficient to prove even this assumption, because in real vortical flows these two terms may have the same order of magnitude, so that their possible contribution to the far acoustic field may be comparable. Anyway, this question can be solved only if one operates with the correct formulae for aerodynamic sound sources, but not equation (45).

Expression (58), which contains vorticity, was interpreted by Powell as a "dipole-like sound source", although all these manipulations have been made only within the right-hand side of equation (45). So this version of Lighthill's acoustic analogy has launched the misleading idea of connecting the sound sources solely with the region of non-zero vorticity, at least in subsonic near-homentropic flows. Qualitatively, this may hold true somewhere, because strong instability effects may take place in the flow region where vorticity is concentrated. Indeed, the background-flow structure there is able to change rapidly, and in turn this gives rise to sound generation. But this scenario remains a coarse hypothesis until an accurate theory is available to define the aerodynamic sound sources. Powell's idea evoked a lot of further efforts [51, 101, 102] to reduce any problem of sound generation in a subsonic flow solely to the analysis of vorticity evolution within the model of incompressible fluid flow. However, it will be shown later that a radically new approach [60–62] gives a quite unusual solution to this problem.

Ribner [47] applied a new variable $p_\omega = p_\varepsilon - p_\beta$, which was groundlessly interpreted as the "acoustic pressure", to the modifying of equation (45) in the case of subsonic homentropic flow without external sources. Then a new form of equation (57) was obtained,

$$\frac{1}{a_0^2}\frac{\partial^2 p_\omega}{\partial t^2} - \Delta p_\omega \approx Q_\omega, \tag{59}$$

where the term Q_ω, called a "monopole-type sound source", was written as

$$Q_\omega = -\frac{1}{a_0^2}\frac{\partial^2 p_\beta}{\partial t^2}.$$

Though this model possesses evident flaws resulting from approximation (57), it is curious in that the term Q_β does not depend explicitly on the velocity of an incompressible fluid, and temporal changes in the only scalar variable p_β are decisive. In an oblique manner, this thereby contradicts the common opinion [49–51] that unsteady irrotational flow is unable to yield volume sound sources (see also section 4.4). Meanwhile, even without discussing the validity of equation (59), Galilean non-invariance of both its sides leads to serious ambiguities when one tries to apply it to the study of many real flows, for instance, jets with convected vortical structures.

This approach was often criticized, so that it is less popular among acousticians. For instance, in reference [51] the assertion has been made (later that was also repeated by Crighton in his review [97]) that the total instantaneous source strength (evaluated by integrating Q_ω over all the unbounded domain G_∞) is infinite on the ground only that the variable part of p_β decreases as $|\mathbf{r}|^{-3}$ when $\mathbf{r} \to \infty$. However, even rejecting this model as a whole, one cannot agree with this conclusion because it is at variance with the elementary mathematical rule: for a function p_β the equality

$$\int_{G_\infty} \frac{\partial^2 p_\beta}{\partial t^2}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3 = \frac{\partial^2}{\partial t^2}\int_{G_\infty} p_\beta\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

is valid only if both integrals on the left-hand and right-hand side exist (are uniformly convergent). It is possible that the left integral, which defines the total strength of Q_ω, is convergent when the integral of p_β is divergent. Generally, the behavior of Q_ω at infinity, however, demands much more accurate analysis. Anyway, this approach still continues the mistaken line of Lighthill's acoustic analogy in which a separate scalar "inhomogeneous wave equation" is supposed to be sufficient for the definition of aerodynamic sound sources, and where no idea has been given for the accurate separation of acoustic components.

7.2. PHILLIPS' AND LILLEY'S FORMS OF HIGH-ORDER EQUATIONS

In further development of Lighthill's method, two approaches have been proposed by Phillips [48] and Lilley (e.g., a version considered in reference [72]), mainly to deal with high Mach number flows, because the authors believed that the classical D'Alembertian on the left-hand side of equation (45) did not conform to the pronounced convective effects in such flows.

Following reference [48] one can introduce a new variable $\beta = \gamma^{-1} \ln(p/p_0)$, $p_0 = \text{const}$. Then equations (1) and (2) can be written as

$$\partial \mathbf{u}/\partial t + (\mathbf{u}, \nabla)\,\mathbf{u} + a^2\,\nabla\beta = \mathbf{f}, \tag{60}$$

$$\nabla\mathbf{u} + \mathrm{d}\beta/\mathrm{d}t = \zeta, \tag{61}$$

where $\zeta = q/c_p + \xi$, $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{u}, \nabla)$. Then applying the divergence operator to equation (60), as well as $\mathrm{d}/\mathrm{d}t$ to equation (61), and taking the difference, one obtains the second-order equation

$$\mathrm{d}^2\beta/\mathrm{d}t^2 - \nabla(a^2\,\nabla\beta) = \nabla[(\mathbf{u}, \nabla)\mathbf{u}] - (\mathbf{u}, \nabla)(\nabla\mathbf{u}) - \nabla\mathbf{f} + \mathrm{d}\zeta/\mathrm{d}t. \tag{62}$$

One can also write in tensor form that

$$\nabla[(\mathbf{u}, \nabla)\mathbf{u}] - (\mathbf{u}, \nabla)(\nabla\mathbf{u}) = (\partial u_j/\partial x_i)/(\partial u_i/\partial x_j),$$

and so equation (62) can be rewritten as

$$\frac{\mathrm{d}^2\beta}{\mathrm{d}t^2} - \frac{\partial}{\partial x_i}\left(a^2\,\frac{\partial\beta}{\partial x_i}\right) = \frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} - \frac{\partial f_i}{\partial x_i} + \frac{\mathrm{d}\zeta}{\mathrm{d}t},$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + u_i\partial/\partial x_i$. In reference [48] this equation was called a "non-homogeneous convected wave equation", and the right-hand side has been interpreted (without any convincing proofs) as a "sound source". However, like equation (45) this equation is not closed, and the second-order differential expressions on both right- and left-hand side contain unknown *total* variables. Thereby, the basic concept of reference [45] was followed again: a convenient-looking form of the "wave propagation operator" on the left-hand side of a certain scalar second-order equation was the main reason for defining the remaining part on the right as a "source term". Thus, this equation cannot give any consistent way to define even a sole component of the aerodynamic sound source $\mathbf{Y}_s$, though being formally compared with equation (45), it shows a number of "advances": both its sides are Galilean invariant (because the Galilean-invariant operator $\mathrm{d}/\mathrm{d}t$ was applied to equation (61)), it contains no ambiguous constants like a_0 in equation (45), etc.

By applying operator $\mathrm{d}/\mathrm{d}t$ to both the parts of equation (62), Lilley derived a third order equation, where a new wave operator on the right was presumably intended to describe better the phenomena of sound propagation in shear flows, and again the right-hand side was treated as a "sound source". In the author's opinion, this "new stride" actually makes no essential advance in comparison with Phillip's equation.

So both these equations, as well as equation (45), are unlikely to be really helpful in studying the phenomena of sound generation by flows. Indeed, each of them represents the dependent scalar equation for diverse *total* variables without explicitly separating out acoustic disturbances, and so all these equations form merely a kind of supplementary to the basic system (1)–(4). Nevertheless, the continuation of this approach can be found in current efforts to simulate the phenomena of sound generation in turbulent flows. In this context, references [4, 103] should be mentioned among the recent works in theoretical and computational aeroacoustics. In the special Appendix of reference [4] (by the way, this paper, as a key example, has been cited by Crocker in section 3, "Computational Aeroacoustics", of his paper [104]), a kind of scalar "convected wave equation" (A19)–(A20) has been derived by Lilley for the approximation of aerodynamic sound sources in a turbulent flow. The dominant acoustic source was there assumed to be determined by the large-scale vortical motion which develops on the background of unbounded steady subsonic parallel mean flow. Actually, the equation for fluctuations $\mathbf{Z}_\varepsilon$ was obtained by applying the procedure of time averaging to Phillips' equation (62) which was regarded as a valid basis. However, no comprehensive description of the time-averaging procedure has been given (interval τ was there merely supposed to be large compared with the characteristic time of large-scale vortices). As a result, the right-hand second-order differential expression, which contained the unknown variables $\{p_\varepsilon, h_\varepsilon\}$, was treated as a sound source. We have previously discussed similar approaches, and all their flaws have been analyzed. A few concluding phrases from that paper acknowledge the habitual concept applied in many other studies: "Equation (A19) is not unique in respect of the noise generation from a given turbulent flow … In some flows this is clearly not best choice … However since the data-base for the time-dependent flow will be the same, irrespective of the choice made for the equations to resolve the acoustic field, the sound radiation to the far field must be independent of the choice of flow variables and their equations". No one is able to dispute this assertion, but these phrases, like the injunctions for future research, by no means can justify the validity of this approach to the definition of aerodynamic sound sources.

The Reynolds-averaged equations supplemented with the standard two-equation model of small-scale turbulence were applied in reference [4] to obtain a computational solution for a certain flow field, which was further used in estimating the sound sources within this version of Lighthill's acoustic analogy. The main conclusion is however modest: "By this procedure the radiated total acoustic power is found to be of the *correct order of magnitude*". So the question remains whether the same result can be obtained with less effort by applying dimensional analysis.

Anyway, the above method has been accepted by some acousticians. For instance, the following was declared in reference [36]: "But the most satisfying technique seems to be the one developed by Lilley, who dealt with this issue by deriving an inhomogeneous convected wave equation for the sound propagation in a transversely sheared mean flow. Most successful noise-prediction techniques (at least in the United States) are now based on the high-frequency solution to this equation".

7.3. HOWE'S APPROACH

The way proposed by Howe [50] should be considered separately because it departs substantially from the above approaches, and so it is often regarded as a fundamental extension of Lighthill's model. It is known that in an irrotational homentropic flow without any external sources one has the non-linear equation

$$\Re B = 0, \quad \text{where } \Re = \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\right] + \frac{1}{a^2}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}\nabla - \Delta. \tag{63}$$

Here $B = \mathbf{u}^2/2 + h$ is the stagnation enthalpy, and $\mathrm{d}/\mathrm{d}t = \partial/\partial t + (\mathbf{u}, \nabla)$. For such a flow one can write

$$\nabla(B + \partial\varphi/\partial t) = 0, \quad \mathbf{u} = \nabla\varphi,$$

and so the following equation is valid as well:

$$\Re(\partial\varphi/\partial t) = 0.$$

Thus, Howe has interpreted equation (63) as the "homogeneous convected wave equation" which is responsible for the propagation of irrotational *acoustic* disturbances in an irrotational homentropic mean flow. However, an important correction should be made: operator $\Re$ is related to the evolution of all kinds of irrotational perturbations, not only those of acoustic waves, in an irrotational homoentropic flow.

In the more general case when gradients of both vorticity and entropy are present in inviscid gas flow without external sources and forces, Howe suggested to take the stagnation enthalpy as "the main variable" for which he derived the new equation

$$\Re B = \nabla(\boldsymbol{\omega} \times \mathbf{u} - T\nabla s) - \frac{1}{a^2}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}(\boldsymbol{\omega} \times \mathbf{u} - T\nabla s) \tag{64}$$

where the right-hand side was *assumed* to be regarded as an acoustic source and $\boldsymbol{\omega}$ is the vorticity. Clearly, the right-hand side of equation (64) reduces to zero in an irrotational homoentropic flow, and this fact was the major argument in favor of such an assumption. An additional argument was also given: if $s = \text{const}$ and $M_f \ll 1$, then equation (64) reduces to Powell's result, although the latter is far from being a standard of accuracy.

Of course, equation (64) is quite correct since it has been derived by exact transformations of the basic equations of fluid mechanics. However, let us discuss the result. This scalar equation cannot be considered separately because the set of *total* variables $\mathbf{u}$, p, s are present there. No convincing arguments can be found why this equation represents an essential advantage over the usual equations of fluid mechanics (1)–(4) if it is applied to the problems of aerodynamic sound. Generally, it is impossible to clear up which terms in equation (64) should be attributed solely to the processes of sound generation by flow, since no way has been given how to distinguish between the acoustic field and unsteady background flow. Actually, as in Lighthill's approach, a number of non-linear terms have been transferred into the right-hand side with the aim to call the latter as a sole sound source, and the "good appearance" of the resulting left-hand side (that may contain something like an extended wave-propagation operator) is presented as the only justification for this conclusion. Even the fact that this "sound source" has non-zero value in a steady flow, embarrasses no one. So this equation, as well as equation (45), cannot give any idea for specifying *each* component of the possible sound source $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, q_s\}$ within a certain *closed* system of acoustic equations.

From the general system (1)–(4) one can derive an exact second-order equation which may be regarded as an extension of Howe's equation. If the relation $\rho^{-1}\nabla p = \nabla h - T\nabla s$ is used, then equation (1) can be rewritten as

$$\partial\mathbf{u}/\partial t + \nabla B + \mathbf{L} = 0, \quad \mathbf{L} = [\boldsymbol{\omega} \times \mathbf{u}] - T\nabla s - \mathbf{f}. \tag{65}$$

Here $\mathbf{L}$ may be treated as an extended version of Lamb's vector. Also, one can write

$$\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t} + \mathbf{u}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\mathrm{d}h}{\mathrm{d}t} + \mathbf{uf} - \mathbf{u}\nabla h + T\mathbf{u}\nabla s = \partial h/\partial t + \mathbf{uf} + T\mathbf{u}\nabla s.$$

Since $\partial h/\partial t = \rho^{-1}\partial p/\partial t + T\partial s/\partial t$, then

$$\frac{\mathrm{d}B}{\mathrm{d}t} = T\frac{\mathrm{d}s}{\mathrm{d}t} + \mathbf{uf} + \frac{1}{\rho}\frac{\partial p}{\partial t} = Tq + \mathbf{uf} + \frac{1}{\rho}\frac{\partial p}{\partial t}.$$

From equation (65) one derives the equation

$$\Delta B = -\nabla \mathbf{L} - \frac{\partial \nabla \mathbf{u}}{\partial t} = -\nabla \mathbf{L} - \xi - \frac{1}{c_p}\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}.$$

By using the above relations, now one can compose the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a^2}\frac{\mathrm{d}B}{\mathrm{d}t}\right) - \Delta B = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}\right)$$
$$+ \frac{1}{c_p(\gamma-1)}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{1}{c_p}\frac{\partial}{\partial t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{uf}}{a^2}\right) + \nabla \mathbf{L} + \frac{\partial \xi}{\partial t}.$$

But it should be taken into account that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\gamma p}\frac{\partial p}{\partial t}\right) - \frac{\partial}{\partial t}\left(\frac{1}{\gamma p}\frac{\mathrm{d}p}{\mathrm{d}t}\right) = -\frac{\nabla p}{\gamma p}\frac{\partial \mathbf{u}}{\partial t} = \frac{\nabla p}{\gamma p}(\mathbf{L} + \nabla B).$$

Thus, the final form of the second-order equation is written as

$$\Re B = \frac{1}{c_p(\gamma-1)}\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} + \frac{1}{c_p}\frac{\partial}{\partial t}\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{uf}}{a^2}\right) + \nabla \mathbf{L} + \frac{\mathbf{L}\nabla p}{\gamma p} + \frac{\partial \xi}{\partial t}$$
$$= \frac{1}{c_p(\gamma-1)}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{c_p}\frac{\partial q}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{uf}}{a^2}\right) + \nabla \mathbf{L} + \frac{\mathbf{L}\nabla p}{\gamma p} + \frac{\partial \xi}{\partial t}, \quad (66)$$

where the operator $\Re$ is now defined as

$$\Re = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\right) - \left(\frac{\nabla p}{\gamma p}\right)\nabla - \Delta. \quad (67)$$

Now one can derive an approximate version of this equation by applying the model of globally compressible subsonic flow [86] where system (47)–(50) is used. Within that model a solution $\mathbf{Z}_\tau(\mathbf{r}, t) = \{\mathbf{u}, p, P, \rho, s\}$ can be obtained to the definite initial-boundary-value problem posed in $G \times J_t$. Further, one can find the density correction ρ_1 from the equation of state

$$\Im(s, P+p, \rho+\rho_1) = 0 \quad \text{or} \quad \frac{P+p}{(\rho+\rho_1)^\gamma} = \frac{P}{\rho^\gamma} \quad \text{or} \quad \rho_1 \approx p a^{-2}, \quad a^2 = \frac{\gamma P}{\rho}.$$

So here one has $\|\rho_1/\rho\| = O(\varepsilon)$, where the small parameter ε is estimated by $\|p/P\|_G$.

By using the solution $\mathbf{Z}_\tau(\mathbf{r}, t)$, one can approximate equation (66) (the omitted terms are of higher orders) as

$$\Re_\tau B \approx \frac{1}{c_p(\gamma-1)}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{1}{c_p}\frac{\partial q}{\partial t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{uf}}{a^2}\right) + \nabla \mathbf{L} + \frac{\mathbf{L}\nabla p}{\gamma P} + \frac{\partial \xi}{\partial t} \quad (68)$$

where

$$\Re_\tau = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{a^2}\frac{\mathrm{d}}{\mathrm{d}t}\right) - \left(\frac{\nabla p}{\gamma P}\right)\nabla - \Delta, \quad a^2 = \gamma P/\rho. \quad (69)$$

Further, a relevant equation (non-linear or linearized) for the disturbances of the stagnation enthalpy can be derived with the use of a certain procedure of time averaging. However, in contrast to equation (14) from reference [105], these disturbances are able to describe many phenomena except sound.

In the particular case $q = 0$, $\nabla s = 0$, $\mathbf{f} = 0$, $\xi = 0$ this equation reduces to

$$\Re_\tau B \approx \nabla \mathbf{L} + (\gamma P)^{-1}(\mathbf{L} \nabla p), \quad \mathbf{L} = [\boldsymbol{\omega} \times \mathbf{u}],$$

and in the case of quite low Mach numbers, when $\| l(\nabla p)(\gamma P)^{-1} \| = O(U^2 a^{-2}) \ll 1$, the right-hand side of this expression reduces to Powell's "sound source"

$$\Re_\tau B \approx \nabla [\boldsymbol{\omega} \times \mathbf{u}].$$

Thus, the main terms in equation (68) are similar to those in equation (66) which was obtained within the general model of compressible fluid flow, and expression (69) resembles something like an extended "sound propagation operator". However, equation (68) is in no way connected with acoustics since in the model of reference [86] all sound waves are characteristically precluded.

A number of examples have been given in sections 5–10 of reference [50] to show the application of this theoretical approach in diverse practical problems. Obviously, if a wrong theoretical model has been suggested for the general definition of aerodynamic sound sources, it is of no use to discuss any examples of its application. Nevertheless, in the author's opinion, some of these examples should be mentioned since they have led to false conclusions about the character of some important physical processes, and these conclusions have been further cited by many.

For instance, in section 5 the well-known solution was taken when an elliptic zone with uniform non-zero vorticity is rotating with constant angular velocity in a two-dimensional irrotational flow of incompressible fluid (evidently, the same irrotational flow takes place in a fluid medium outside the rotating elliptic cylinder). Then the right-hand side of equation (64) was treated as the sound source $Q \approx \nabla[\boldsymbol{\omega} \times \mathbf{u}]$ in a "similar" homentropic subsonic flow of compressible fluid (although no one has succeeded in finding an exact solution for such a "similar" flow), and the known solenoidal velocity field $\mathbf{u}_\beta$ taken from that solution was used to approximate this "sound source". As a result, the following conclusion was expressed: this "similar" subsonic flow causes the quadrupole-type sound emission. However, if that exact solution is used as basic for the approximation of sound sources defined within the radically new theory [60–62], such a flow will not radiate sound (this flow will be discussed in detail later along with the description of the new theory). So the spurious "quadrupole-type sound source" arose in reference [50] only due to the wrong theoretical model being applied.

Also, in section 7 the generation of sound by entropy inhomogeneities was considered. With attention confined to low Mach number flows, and with the contribution of both vorticity and heat conduction to the sound radiated being neglected, equation (64) was there assumed to have the form

$$\Re B = -\nabla (T \nabla s). \tag{70}$$

However, these thermal phenomena can be successfully simulated in subsonic flows if one applies the theoretical model [86] in which all acoustic effects are characteristically precluded, and then an approximation of equation (70) with the non-zero right-hand side can be readily obtained from equation (68). So the term $\nabla(T \nabla s)$ should be taken into account when the evolution of unsteady background flow is analyzed, but there is no reason for treating it as a sound source. Generally, this example from reference [50] gave rise to numerous subsequent works, where the moving “entropy spots” were studied as the main sound sources in accordance with Howe’s approach.

The paper by Möhring [51] may be regarded as a further “development” of the Powell–Howe approach. Actually, that work was based on the assumption that in a subsonic homentropic flow at low Mach number the scalar sound source could be defined as in equation (58). There, a few particular incompressible flows were considered (moving vortex rings and compact vorticity spots) with the aim of finding the proper relations between vorticity field and the sound radiated through calculating the Green functions of special form. We have discussed the principal defects of the models given in references [49, 50], and so now we will not analyze the contents of reference [51] where all those mistakes have been inherited. Even the recent attempt of this kind [38] in deriving the “modified” form of equation (64) does not depart essentially from the basic Lighthill–Powell–Howe concept.

The opinion is now universally recognized [16, 97] that all kinds of sound source definitions (monopole, dipole and quadrupole types in the ordinary classification), suggested within all the variety of existing theories, are “equally exact” and differ only in their *non-radiating terms* which produce zero contribution in the far sound field. This assertion is used as a cunning justification for numerous incongruities and contradictions in these approaches. Surely, a definite scalar function $\vartheta(\mathbf{r}, t)$ can be easily investigated as an *externally assigned* source in simplest linear equation (42). But in the general case one should give an accurate definition of the sound source $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, \mathbf{q}_s\}$, caused by a high-unsteady flow, *before* such a source is analyzed. Anyway, one cannot consider a scalar second-order differential expression, which contains the set of unknown total variables, as a radiating or non-radiating sound source, when no adequate procedure of flow decomposition has been found.

7.4. DOAK’S MODEL

Doak [105] used time averaging, over an interval τ which could be selected arbitrarily, to define the fluctuating stagnation enthalpy B_ε (subsequently it will be denoted as B') as “the basic variable for acoustic field“, and then derived “an exact second-order inhomogeneous scalar wave equation of convected type” the left-hand side of which is linear in B'. The newest version of this equation presented in reference [35] includes external forcing as well as both heat and mass additions. In a wide sense this equation develops Howe’s concept, but it is different from the latter in several respects.

The right-hand side, which is supposed to form the sound source, consists of second-order non-linear differential expression including the main variables and their fluctuations (among those the Coriolis acceleration $\mathbf{u} \times \boldsymbol{\omega}$, where $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$), as well as the externally assigned sources. If the fluid in the region concerned is lossless and subject to no externally assigned sources, and it is in a uniform rest state apart from small amplitude, irrotational and homogeneous fluctuations, this equation reduces to the homogeneous D’Alembert wave equation. Since this equation for B' in the general case is exactly valid for a fluid in any state of motion, Doak has claimed that B' can be regarded as “the basic generalized acoustic field”, though a comment has been made that these B'-waves “are not necessarily acoustic waves, or even waves in the classical sense; nevertheless, when the B' disturbances reach a fluid region of otherwise uniform mass density and temperature, and otherwise at rest or in uniform motion, they become identical to classical acoustic waves of fluctuating pressure per unit (constant) mass density”.

In reference [105], however, Doak emphasized a serious flaw of this equation. This is its mathematical "redundancy", generally similar to that which is present in Lighthill's and Howe's approaches. Indeed, $B' = h' + (\mathbf{u}^2/2)'$, and both h and u_i appear in the coefficients of the left part as well as in the terms of the right part. So this equation for B' can be accurately satisfied only if the five independent scalar variables $\{u_1, u_2, u_3, h, s\}$, as well as their disturbances after applying the routine procedure of time averaging, are already known in $G \times J_t$. But these variables could be obtained as a solution to a certain initial-boundary-value problem posed for the *closed* system (1)–(4). Therefore, Doak has suggested some approximate ways in which this redundancy can be mitigated in practice; for instance, the left-hand side of equation for B' may be linearized in the fluctuations irrelevant to the right-hand side.

One can agree that the non-linear term $\mathbf{u} \times \boldsymbol{\omega}$ may be of particular importance in many real flows, but generally it is difficult to accept the common opinion mentioned in reference [105]: "it has become more and more widely acknowledged that this acceleration plays an often leading role in the aerodynamic generation of sound". If one calculates the fluctuation of this term on the background of time-averaged flow (or of "quasisteady flow" if the interval τ is large enough — see section 4), then this value can be divided into two parts: the first one has to be attributed to the high-unsteady non-acoustic motion with non-zero vorticity, and the second one to the acoustic field which may be vortical as well. Now one would recall equation (68) derived within model [86]: there the term $\mathbf{u} \times \boldsymbol{\omega}$ was present too, and so its fluctuations could be defined as well, but all sound waves were characteristically excluded from that model. Of course, one could assess the contribution of various non-acoustic disturbances, including a relevant part of this term, to the components of aerodynamic sound source $\mathbf{Y}_s$, but only if a consistent definition has been found for such a source.

What should be also noted again is that, the separation out of the well-looking left side (even if this side consists of something like "the wave-propagation operator" applied to a certain variable) from a sole scalar second-order equation which usually contains all flow variables, cannot be a sufficient ground for treating the right-hand part, which is the second-order non-linear differential expression, as a "sound source", and this delicate question was discussed in sections 5 and 6.

Thus, though the reference [105] represents the most comprehensive analysis in this specific direction where the stagnation enthalpy is regarded as the dominant variable, Doak's scalar second order equation for B' cannot serve as a rigorous mathematical model for defining all components of the sound source $\mathbf{Y}_s$. Nevertheless, this equation may be of use when linearized in the fluctuations for some approximate calculations of acoustic/mean-flow interaction effects.

7.5. ON THE DEFINITION OF ACOUSTIC ENERGY

A lot of efforts have also been aimed at deriving an approximate equation for the energy E_α of acoustic disturbances. Various approaches to this problem have been described, e.g., in references [2, 106–108], and there is no need to make a detailed review again. Perhaps an additional analysis of the energy balance could be useful within a certain linear model, but the known efforts in deriving an appropriate energy equation cannot be regarded as quite successful, although in some particular cases the proposed versions of such an equation may be valid (e.g. in the geometric acoustics approximation). The main problem, mentioned in reference [2], is that the second-order term in the expansion for E_α would contain not only the products of linear perturbations of flow variables, but also second-order perturbations of the variables; these latter are however beyond the scope of any linear model. Besides, the pronounced flaws of the existing aeroacoustic models, which have been discussed above, have led inevitably to the related errors in deriving an accurate "acoustic energy equation". For instance, if one tries to separate out the small disturbances $\mathbf{Z}_\varepsilon(\mathbf{r}, t)$ on the background of "quasisteady flow" (e.g., by applying a procedure of time averaging such as that considered in section 4), these disturbances will contain not only sound, but also the fluctuations of both vorticity and entropy, and then a possible definition of energy for such disturbances cannot be attributed solely to sound waves. All these problems explain why the general definition of E_α in a complex unsteady non-uniform flow still remains ambiguous.

Evidently, the most general *non-linear* equation for the acoustic energy E_α with an adequate sound source, as well as the rigorous definition of E_α, can be readily found after creating a *closed non-linear* system which would describe the evolution of acoustic disturbances separated out on the *unsteady* background flow. But no well-known approach in aeroacoustics suggests such a general solution.

Suppose a procedure of decomposition $\mathbf{Z} = \mathbf{Z}_v + \mathbf{Z}_\alpha$ has been made within the general concept given in section 4.5, and, as a result, a closed system of non-linear equations for the background-flow variable $\mathbf{Z}_v(\mathbf{r}, t)$ has been derived (although this is a very complex problem, and so now it is not discussed in detail). Then taking the background-flow variable $\mathbf{Z}_v(\mathbf{r}, t)$ as a known function, one can readily obtain a closed system for the acoustic variable $\mathbf{Z}_\alpha(\mathbf{r}, t)$, that complements the $\mathbf{Z}_v$-system to the basic system (1)–(4). Within the so-found acoustic system one can operate a minimal number of independent variables, for instance $\mathbf{Z}_\alpha = \{\mathbf{u}_\alpha, p_\alpha, \rho_\alpha\}$, and so one can do without any additional *dependent* variable like E_α. But if one intends to operate it, the derivation of the relevant non-linear equation will not represent a difficult problem. Note that if a linearized version of this acoustic system is used, the keen question may arise again whether this variable can be well conformed to the linear solution $\mathbf{Z}_\alpha$.

Thus, first one should solve a certain initial-boundary-value problem for unsteady background flow. Then, taking the solution $\mathbf{Z}_v(\mathbf{r}, t)$ as a known function in $G \times J_t$, one can write the exact energy equation which in the absence of external source terms in the basic system (1)–(4) will have the form

$$\partial E_v/\partial t + \partial E_\alpha/\partial t + \nabla \mathbf{N}_v + \nabla \mathbf{N}_\alpha = 0,$$

where

$$E_v = \rho_v(e_v + \mathbf{u}_v^2/2), \quad \mathbf{N}_v = \mathbf{u}_v(E_v + p_v), \quad e_v = e(p_v, \rho_v) = p_v[p_v(r-1)]^{-1},$$

$$E_\alpha = \rho_\alpha(e_v + \mathbf{u}_v^2/2) + (\rho_v + \rho_\alpha)(e_\alpha + \mathbf{u}_v \mathbf{u}_\alpha + \mathbf{u}_\alpha^2/2),$$

$$\mathbf{N}_\alpha = (\mathbf{u}_v + \mathbf{u}_\alpha)(E_\alpha + p_\alpha) + \mathbf{u}_\alpha(E_v + p_v), \quad e_\alpha = e(p_v + p_\alpha, \rho_v + \rho_\alpha) - e_v.$$

This equation can be written in another form with the "energy source" j_v which is determined solely by the solution $\mathbf{Z}_v(\mathbf{r}, t)$:

$$\partial E_\alpha/\partial t + \nabla \mathbf{N}_\alpha = j_v = -\partial E_v/\partial t - \nabla \mathbf{N}_v.$$

Of course, diverse simplified and linearized versions of this equation, applicable to particular flows, may be further derived. But in any case the quite general definition of acoustic energy in a high-unsteady flow seems to be fully dependent on the solution of the key problem in aeroacoustics that implies an adequate decomposition of each flow variable into the sound disturbance and the non-acoustic component attributed to unsteady background flow.

7.6. KIRCHHOFF'S THEOREM

It is appropriate to mention the widely used method for the farfield sound prediction, which is based on Kirchhoff's theorem; among the early works on this one can note Blokhintsev's monograph [2]. Farassat and Myers [53] have extended this method to the sound radiation from moving surfaces, even with the presence of supersonic zones of the flow [56]. In more recent works [57, 58], some correction terms have been suggested to reduce the volume of the computational domain. So various versions of the method are rather popular presently.

According to that general approach, pressure and pressure derivatives, both in time and along the normal (hypothetically these may be calculated from the previously obtained nearfield CFD solution), are assigned on a surface Θ that encloses the local region G_f with sound sources: for instance, rotor blades. In the derivation of Kirchhoff's theorem the rather strong assumption is used that only linear acoustic propagation occurs outside the integration surface, and all acoustic sources and non-linear effects are contained within that surface. Thereby, this approach represents a kind of spatial decomposition where the flow region with intense sound sources and the "external" sound field, produced by these latter, are completely separated.

Unfortunately, the calculation of unknown distributions on that surface Θ is equivalent to the solution of the whole problem for the explicitly separated acoustic variables $\mathbf{Z}_\alpha(\mathbf{r}, t)$ inside G_f with *non-reflecting* boundary conditions to be assigned at the boundary surface Θ. But it seems scarcely probable that one can pose such a difficult initial-boundary-value problem within G_f, primarily because of the absence of an adequate theory of aerodynamic sound in a high-unsteady flow, and so a coarse approximation is usually used for those distributions. Even if one carries out the computational simulation of unsteady flow in a finite domain G_f by applying the general non-linear equations (1)–(4), it may be inaccurate to take the whole boundary as an integration surface Θ since it is hardly possible to assign the perfect sound-absorbing boundary conditions on all boundaries crossing the flow; besides, it is not trivial to derive the necessary distributions along Θ from a certain solution $\mathbf{Z}(\mathbf{r}, t)$ obtained in terms of total variables. One may try to simplify this problem by moving Θ far away from the flow region with sound sources, so that the *whole* surface Θ would cross the quiescent gas medium; but all the same, in most practical cases the surface Θ will contain both inflow and outflow parts, and generally any increase in the volume of computational domain will lead to substantial penalties in the computer time. Also, the sound propagation over a long distance may result in substantial effects of spurious dissipation and dispersion which are inherent in the computational code, especially within the short-wave band. So the optimal choice of surface Θ is the crucial point of this approach when the computational solution $\mathbf{Z}(\mathbf{r}, t)$, found in a finite spatial domain, is used in an attempt to estimate the far sound field which would take place in an infinite domain.

Anyway, one should remember that this theorem results from the simplest linear acoustic equation written for a *single scalar variable*, although very few aeroacoustic problems can be reduced to this particular case. In its key conclusion this theorem shows that a definite configuration of the far sound field can be produced not only by true volume sources, but also appropriate non-unique distributions of spurious sound sources on Θ are able to give the same result. Hence, one would agree with the opinion expressed in section 1.5.1 of reference [72] that this theorem may be applicable only for the *qualitative* analysis of sound propagation in the far field ("for analyzing the farfield propagation of the previously specified sound waves" seems more correct), and by no means can it reveal the mechanism of sound generation by unsteady flow.

8. ON EXPERIMENTAL AND COMPUTATIONAL RESEARCH IN AEROACOUSTICS

The above analysis of the existing "purely theoretical" approaches in aeroacoustics will be incomplete if it is not supplemented by some comments on the main problems in experimental and computational investigation of the key aeroacoustic phenomena. However, if one is waiting for an exhaustive review of this enormous field, that is not what can be done in this section. The main purpose is to discuss a few examples in which some experimental and computational results have been drawn in efforts to confirm the validity of the above-mentioned approaches to the theory of aerodynamic sound. In this aspect the inherent restrictions of experimental research should be emphasized. Also, it is worth showing the urgent need for developing much more efficient mathematical models and algorithms for the numerical simulation of most unstudied aeroacoustic phenomena, particularly the processes of sound generation by a high-unsteady flow, because most of the current computational methods are imperfect. Anyway, the computational methods, which are based on a series of mathematical models, should be regarded as an inseparable part of the whole scope of theoretical methods in aeroacoustics, and so it would be illogical to pass over this vast area. Moreover, one cannot avoid such a consideration when following the major idea of this work to select the most serious flaws in the present theoretical fundamentals of aeroacoustics.

8.1. EXPERIMENTS

One can now comment briefly on the numerous experimental results which are often quoted to confirm the validity of the basic Lighthill model and the other approaches mentioned above. From diverse works one can easily draw the fundamental conclusion that no acoustic measurements, made in the far field, can reveal in a unique manner the distribution of sound sources within the local flow region where sound is generated. So no farfield probes are likely to be sufficient for the study of the local mechanism of sound generation. On the other hand, one will introduce a lot of disturbances into the flow by inserting inside it even the smallest acoustic probes, and thereby one may change drastically all the flow properties. Moreover, it seems absolutely impossible to estimate accurately all components of the possible sound source $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, q_s\}$ just in the region with a high-unsteady flow.

Here it is relevant to cite a few introductory phrases from reference [109]: "Measurements in the far field, no matter how detailed and sophisticated, cannot lead to a unique picture concerning the nature of the acoustic sources. One is forced therefore to make measurements at the source location as well. This, however, proves to be a most elusive task. Since the noise production is associated with a volume integral, point measurements (or even two- or three-point correlation measurements) are insufficient to lead one to the desired picture of the sources". As another example, in section 7.2 of reference [90] one can read: "It is no good listening to a sound, or even analyzing its structure with the most sophisticated techniques and equipment, if the aim is to describe its source with certainty. That aim is not realizable".

Thus, the absence of reliable experimental data, which would give the distribution of sound sources in unsteady flow, prohibits the verification of any aeroacoustic theory. This seems to be the main reason why different theoretical models of sound generation, even if these are evidently invalid, still exist. Nevertheless, new experiments (e.g., where sound is radiated by colliding vortex rings [110, 111]) were carried out in an attempt to justify Lighthill's model.

Suppose that one has found a "satisfactory agreement" between an experiment and a noise prediction made with the use of a certain aeroacoustic theory. However, this cannot be a decisive argument for accepting immediately that theoretical model as the best approach. To this should be added that such an agreement may be only partial (typically, in approximately comparing the noise intensity *in the far field*), since no one possesses the proper equipment to make all *quantitative* comparisons, much less in the near field. One should also remember that some of the above theoretical approaches result from exact transformations of the basic nonlinear equations of fluid mechanics, and so some phenomena may be reflected *integrally* if the *previously obtained* exact solution $\mathbf{Z}(\mathbf{r}, t)$ of system (1)–(4) is substituted into a scalar second-order equation like equations (45), (62) or (64). Moreover, one may even assume that in the particular flow conditions a certain group of terms on the right-hand side of such a scalar equation may yield a magnitude which is near, at least in its order, to the *dominant* component of the true sound source. The differential expression on the right, defined as a "sound source", can be formally written as a sum of terms, each of them looking responsible for the definite physical effect contributing to sound emission (some examples of such an "analysis" could be mentioned), and generally it is quite possible that a certain term from this sum may influence substantially the mechanism of sound generation, although in no connection with the model suggested. But all the same, any "dominant *scalar* source" will have very ambiguous relation with the general sound source $\mathbf{Y}_s$. As a simple analogy, the known function ϑ in equation (42) is insufficient to restore all components of the source $\mathbf{Y} = \{F_1, F_2, F_3, \zeta\}$, and so the whole solution $\mathbf{Z}(\mathbf{r}, t) = \{\mathbf{u}, p\}$ cannot be found. Thus, no possible coincidence with experiment can serve as a serious ground which is able to justify the "traditional" definitions of aerodynamic sound sources, much less if all inherent defects in these definitions have been revealed above. As an anonymous author said, "the correct theory should give correct results, but the wrong theory is able to produce anything".

Probably, in the future, new devices will be able to give the values of all variables of high-unsteady flow, at any moment as well as at every point, without introducing any disturbance. However, even if one obtains such a perfect facility, perhaps with the use of advanced laser techniques for flow measurements, only the total values of flow variables will be measured, and the key problem will remain: the acoustic disturbances, which have usually very small amplitudes, should be separated out on the high-unsteady background flow at each point of the flow region. So in any serious experiment one cannot do without a consistent theory of aerodynamic sound which provides general definitions of unsteady background flow as well as of both sound waves and sound sources.

Goldstein [72] and Müller [112] have criticized, although in a rather gentle manner, the disparities between the theoretical results obtained with the use of Lighthill's acoustic analogy and some experimental data. For instance, in reference [112] it was shown that some experiments displayed drastic differences in the dependence of the intensity I_s of sound radiated by subsonic jets on the power of characteristic Mach number M of jet flow. Indeed, at low M (i.e., just when the Lighthill's approach aspires to be the most appropriate) one can find that the well-known law $I_s \sim M^8$ is unlikely to be valid. At the same time, no doubts have been expressed there about the basic idea of Lighthill's model. By the way, the laws $I_s \approx kM^6$ and $I_s \approx kM^8$ for the dipole and quadrupole sources in diverse flows were first revealed by Blokhintsev in reference [2], long before Lighthill's works [45, 46] appeared. Therein, it was first demonstrated that dimensional analysis, supplemented by the simplest physical model of a definite flow, was quite sufficient to derive those laws without any plausible theory of sound generation. However, different experimental data, which may qualitatively conform to these laws, give the values of coefficient k ranging within a few orders [72]. Besides, many facts show that the integral intensity of sound generated by jet flow depends not only on the value of the characteristic velocity, but many other factors can be decisive (e.g., the structure of small-scale turbulence at the initial section of a jet [83]). Anyway, one should not regard the rough approximation $I_s \approx kM^8$, that may be well obtained on the basis of dimensional analysis, as a triumph of Lighthill's acoustic analogy.

Thus, any authentic case of "complete agreement" between the theoretically or computationally predicted level of noise emission and the relevant experiments does demand a more precise analysis in order to understand all reasons for this surprising fact.

8.2. COMPUTATIONAL PROBLEMS

A substantial contribution to the comprehension of aeroacoustic phenomena could be made by applying modern computational methods. Presently, a lot of diverse computational approaches are suggested for the simulation of unsteady gas flows. Unfortunately, only a very few of those can be really applied to the solution of complex aeroacoustic problems, much less to the analysis of sound generation in a high-unsteady flow. It seems unreasonable to review here in detail the current state of computational aeroacoustics, including comparisons of the most popular methods as well as a consideration of many definite solutions with the analysis of principal errors in each case (surely, a lot of these can be found). Generally, it is impossible to make such a review within this paper because of its limited length, and this remains a vast theme for prospective publications. Nevertheless, it is worth briefly mentioning a number of key directions in this topical research area.

Perhaps the best situation is found in the computational study of sound propagation in steady subsonic near-parallel flows, both internal and free, where vast experience has been gained in the solution of linear equations for small disturbances, although serious difficulties may arise if a general multidimensional initial-boundary-value problem is posed for the separate study of sound waves (see section 4.2). An additional number of keen questions will appear if any sound sources in the volume, especially those generated by the flow itself, take place. So primary attention will be further focused on the solutions of most difficult problems with sound generation.

If one considers the relevant reviews, for instance those presented in references [11–14, 16, 17, 20, 29, 40], it will be revealed that the existing computational codes, applied for the prediction of aerodynamic sound emission, are based on a small number of the theoretical approaches analyzed above. Since these latter possess too many defects, this causes serious doubts about the accuracy of computational solutions.

For instance, the Ffowcs-Williams and Hawkings equation, as "the most general form of Lighthill's acoustic analogy", is very popular among aeroacousticians [11–13, 20, 27, 40, 98–100]. Also, one can see that Kirchhoff's theorem [13, 14, 53–58] (see section 7.6) is often used for predicting the farfield noise, and this prediction is usually based on a certain CFD-solution previously obtained in a finite domain with a high-unsteady flow.

The typical approach consists of two stages. First, in the stage of "direct numerical simulation" (DNS) one obtains a computational solution to a certain unsteady flow by integrating the non-linear systems of Navier–Stokes or Euler equations (normally in terms of total variables). Generally, the relevant system for the compressible medium may be used [18, 41], but even the model of *incompressible* fluid is often applied in this stage to subsonic flows [6, 31]. In the second stage this solution is taken as an approximation of unsteady background flow, and then a version of Lighthill's acoustic analogy (i.e., the basic equation (45), a modified equation like (62), or the low Mach number equations (57)–(58)) is used to estimate the "sound source" Q_L, and in turn the farfield noise. It seems that the

Powell–Howe formulation has not been applied extensively to such solutions, although Möhring's version of this approach has been used in some works (e.g., reference [31]). All the progress in this computational direction is usually appreciated as quite successful. Nevertheless, in some analytical works one can find a cautious criticism, e.g., in reference [11]: "Though the acoustic analogy has proven to be very powerful for acoustic prediction, many problems of interest do not seem well-suited to formulation in terms of separate source and propagation regions."

As an alternative method [25], the non-linear equations for unsteady disturbances are written on the background of steady mean flow, the latter having been calculated previously by using a separate algorithm. Unfortunately, these disturbances are often treated as sound, although in fact they include all kinds of waves, but not solely sound. Surely, if one takes a quite sizable spatial domain with boundaries far enough from the flow region which is mostly responsible for the generation of these disturbances, hypothetically it is possible to separate out the "far field" sound waves on the background of quasi-steady mean flow at the remote boundary, but practically it is hardly possible to provide the simultaneous co-existence of both near field and far field in any computational model (clearly, this is connected with the usual restrictions in the numerical simulation of extensive flows).

New specific problems arise in the computational study of supersonic flows [4, 43, 113], including supersonic jets [25, 33], where strongly non-linear effects are crucial in the processes of sound generation, and so the full non-linear systems of the Navier–Stokes or Euler equations remain most promising for obtaining a DNS-solution. The small-amplitude sound field may co-exist there with an intricate structure of unsteady shock waves, and then the computational algorithm must meet extremely stringent requirements to provide the necessary accuracy. Clearly, this makes much more difficult the key problem of separating out both the sound waves and sound sources. However, the "traditional" models of aerodynamic sound sources, most of which are based on Lighthill's acoustic analogy, are often suggested for this class of problems (see, e.g., references [40, 114]), although this approach does not hold good even in the much simpler cases of subsonic flows. So the known attempts of this kind do not show encouraging results, even in estimating the total acoustic power output [4]. Anyway, if a computational method, applied to the investigation of sound generation in a high-unsteady flow, is based on a definite mathematical model of aerodynamic sound sources, all defects in the latter will lead to an erroneous final result.

8.2.1. *Direct numerical simulation of unsteady compressible fluid motion*

The rather old but well-founded way may look very attractive: a certain initial-boundary-value problem could be solved by integrating the general systems of non-linear Navier–Stokes or Euler equations for *total* variables $\mathbf{Z}(\mathbf{r}, t) = \{\mathbf{u}, s, p, \rho\}$ which govern the motion of a compressible medium. This approach, which may be classified as a "direct numerical simulation of unsteady compressible fluid flows" (DNSC), seems to be promising for the simulation of any non-linear aeroacoustic phenomena. Formally, it does not need any additional model of aerodynamic sound sources, although a DNSC-solution could represent a database for further application of such a model. Despite some restrictions, this research method possesses a significant advantage over experiment: all the set of flow variables can be obtained at each point and at any moment. Besides, when the means of flow control are looked for, the computational experiment provides a unique opportunity to come back in time and thereby correct the control action which has proved to be ineffective. In this way a tree of solutions, started from the same initial conditions, could be investigated to choose the optimal scenario of flow control. However, the quite accurate solutions of this kind, much less those with the control of non-linear self-excited acoustic phenomena, can be rarely found among scientific publications. When analyzing most known solutions, even if they look quite correct at first sight, one can reveal a number of serious errors either in posing or in solving the relevant initial-boundary-value problem. This is not surprising because many real flows are too complicated to be readily simulated within the existing standard approaches. For instance, in unsteady subsonic flows every local defect of the computational model, either on the boundary or in the volume, is able to exert the drastic influence on the flow evolution in the whole spatial domain due to the global interconnection through sound waves, and then no valid solution can be obtained. So the most accurate approaches, both theoretical and computational, should be applied, and new ones created, to provide an adequate resolution of aeroacoustic phenomena.

When considering this computational approach in aeroacoustics, even without analyzing the definite type of computational algorithm applied for integrating the basic system of differential equations, one can readily emphasize the topical problem of specifying the boundary conditions on permeable or moving surfaces. These conditions should determine the evolution of both unsteady background flow and acoustic field, and in particular they imply the definite local value of normal acoustic impedance according to the notion traditionally accepted in classical acoustics, at least while the boundary is assumed to be locally reacting.

The computational models of inflow and outflow boundaries pursue the aim of minimizing the sound reflection effects (i.e., these boundaries should be perfectly sound-absorbing). Note that the primitive spatially periodic inflow/outflow boundary conditions, often applied in CFD-solutions, are usually inapplicable to aeroacoustic problems. The typical models of non-reflecting boundaries have been developed within a simple linear hyperbolic equation for a single scalar variable [115], or by considering the linearized version of Euler system [116]. Unfortunately, these "characteristic based boundary conditions" may work well only for sound waves which are nearly normally incident on the boundary. More general non-linear models of permeable boundaries, including the non-reflecting ones, have been proposed in references [63–65, 68], and their application by the author in various problems could be appreciated as quite successful, at least until the mean flow near the boundary is "quasi-parallel". However, these approaches do not work effectively in the cases when complex vortex structures take place close to the inflow or outflow boundary. It has been shown in reference [67] that the spurious sound sources on the exit boundary, through which intense vortices escape from the computational domain, in their intensity are able to exceed all the volume sources of sound; this may cause acoustic feedback effects and in turn strong resonance oscillations, etc. — as a result, the flow evolution changes radically. So the specification of a proper system of boundary conditions, which may have to be non-local in both time and space, represents the crucial problem in computational aeroacoustics.

The choice of an adequate finite-difference scheme represents a very important problem as well because one must resolve the phenomena usually featured by the relatively small amplitudes of sound fluctuations. An appropriate scheme should resolve the rather wide range of sound frequencies, and so one must use spatial grids with the minimum mesh size which is much shorter than the minimal wave length to be approximated. Additionally, some specific requirements and restrictions are to be imposed on the structure of a non-uniform spatial grid which covers the finite computational domain; for instance, a substantial grid stretching along some directions is able to cause the spurious effects of absorption, reflection, and refraction in the sound fields.

The processes of sound generation by a flow are essentially non-linear, but most of the frequently applied schemes are unable to provide the promised high-order accuracy in approximating the relevant non-linear equations. Indeed, the order of the scheme accuracy is usually estimated within a certain simplified system of differential equations, often linearized, because generally one cannot do this within the full non-linear system of fluid mechanics equations.

The following requirement should be also emphasized: it is very desirable, especially while simulating subsonic flows, that the spatially symmetrical differences only are used in the computational algorithm (otherwise the related spatial anisotropy may distort drastically the processes of sound propagation); however, this is not a trivial problem. At first sight, the widely applied "upwind approximations" seem much more attractive since they provide the monotone distributions of all variables, but this is attained due to the considerable magnitude of the artificial viscosity inherent in such schemes, that results in strong filtering of high-frequency sound waves. A lot of efforts also have been aimed at minimizing the spurious effects of both dissipation and dispersion in a scheme; evidently that would conform better to the true physics of sound propagation over long distances from the sound source, but one may often agree with the opinion expressed in reference [11] that "this phenomenon is of little significance in typical aerodynamic computations".

Thus, a finite-difference method based on the non-linear Navier–Stokes or Euler systems for compressible media is potentially able to resolve all kinds of wave processes, including sound. But, analyzing the numerous examples of such an approach, including those obtained by the author, one can see that usually it is easy to separate out unsteady background flow (including the convected disturbances of both entropy and vorticity) from the whole computational solution obtained for the total variables. On the contrary, it is very difficult, sometimes hardly possible, to extract the acoustic disturbances from that general solution, especially in internal separated flows, and one encounters a similar problem in experimental research. Nevertheless, this method can be successfully applied to the solution of many non-linear aeroacoustic problems, particularly to the simulation of internal viscous flows where non-linear sound–flow interactions as well as self-excited resonant oscillations take place, as was done in references [75, 76].

Meanwhile, this approach was applied by some in efforts to confirm the validity of Lighthill's acoustic analogy. Among the recent attempts of this kind, reference [41] can be considered. The sound generated by vortex pairing in axisymmetric jets was there determined by direct numerical solution of the compressible Navier–Stokes equations on a computational grid that included both the near field and a portion of the far acoustic field. Concerning section 2 of this work where the method of solution is described, one can notice the absence of any definite specification of the outflow boundary conditions. However, the extreme importance of these latter in the simulation of aeroacoustic phenomena was emphasized above; so no accurate solution of such a problem can be obtained if the wrong outflow conditions are assigned. What is more curious is that in section 4 of reference [41] the so-calculated DNSC-solution was substituted into the right-hand side of Lighthill's equation (45) with the aim of estimating the "sound source" Q_L. Then the far sound field predicted by this "sound source" was compared with the far field (that was supposed to be presented by the fluctuations of div $\mathbf{u}$) obtained directly from the DNSC-solution. Although the conclusion has been there expressed that "these predictions are in good agreement with the directly computed data", the results of those comparisons cause many doubts. Actually Figure 15 of the paper shows that Lighthill's prediction has nothing in common with the DNSC-solution if the "special model for the passive region of the source" is excluded from the procedure of farfield prediction. This model was proposed in reference [41] as a means "for including the effects of axial source non-compactness when calculating the Lighthill predictions", but it was based on the groundless, both physically and mathematically, assumptions on the character of jet evolution in the sizable pre-exit part of the computational domain (a primitive downstream extrapolation was applied to estimate the values of Q_L in that pre-exit zone). The following fact seems to be most contradictory, and so it is able to discredit those results: Figure 15 of the paper shows that after introducing this pre-exit zone one obtains "complete agreement" of Lighthill's predictions with the relevant DNSC-data; thereby this zone, which *does not contain any physically meaningful data*, makes the dominant contribution to sound emission!

Probably, an idea of this pre-exit zone "constructed to allow large-scale vortices to exit the computational domain without reflecting significant acoustic disturbances back into the region of interest", as well as the concept of a "sponge" zone with "perfectly matched layer" applied in references [32, 42], have been borrowed, although in an oversimplified manner, from the fundamental approaches proposed in references [66, 67]. Presumably, if those latter were properly applied in this simulation, one might have no need of designing this highly questionable model.

Also, in section 5 of reference [41] the comparisons were made between the far sound field obtained within a DNSC-solution, and that calculated with the use of Kirchhoff's method applied to a certain integration surface located *inside* the computational domain. As discussed in section 7.6, the choice of an appropriate integration surface is the crucial point of Kirchhoff's approach, and if this surface has been drawn in an optimal manner through the computational domain in which a DNSC-solution was previously obtained, then such a comparison (only in a remote part of the domain, in a medium which is supposed to be quiescent and homogeneous, under the perfectly non-reflecting boundary conditions, etc.) may show a rather good agreement, at least in the long-wave sound band. So the investigation described in that section represents a typical example of such an activity (although under imperfect boundary conditions), and its results do not reveal anything unexpected.

8.2.2. *Turbulence effects*

If one has to resolve the non-linear interactions between large-scale vortical motion and small-scale turbulence (these interactions are able to influence greatly the mechanism of sound generation), this problem poses new, much more stringent, demands on the computational codes. The DNSC-method seems inappropriate for the solution of such specific problems, primarily because the practical sizes of spatial grids are unable to provide the simulation of small-scale turbulence on the basis of the Navier–Stokes or Euler equations. In this case a series of computational methods has been developed for the simulation of unsteady large-scale motion, in a compressible or incompressible medium, the evolution of subgrid scales being taken into account with the use of some additional models. Diverse approaches of this kind are often classified, neglecting possible differences, as the "large eddy simulation" (LES) (see, e.g., references [4, 17, 18, 40, 42, 83]). Some specific versions, for instance that based on the classical k–ε model and called in reference [18] as "semi-deterministic modelling of turbulence" (SDM), or "unsteady Reynolds averaged Navier–Stokes simulations" (RANS) [40] could be also attributed to this family. The original version of this approach has been proposed by the author in reference [83]. If one analyzes these approaches in detail, a lot of sharp questions may arise on the validity of basic assumptions as well as on the ability to resolve *simultaneously* the key acoustic phenomena, especially at high frequencies, and the specific effects of turbulence (note that all details in the time-averaging procedure may be decisive in such problems). Unfortunately, most of these questions cannot be answered in a quite rigorous manner.

Clearly, if the grid size tends to zero, LES may become identical to DNS. In some practical calculations the molecular viscosity and heat conductivity can be ignored in comparison with the related turbulent effects [83]. The contribution of subgrid scales in the sound emission is neglected within LES, although some estimations have been performed in reference [44] to show that this contribution should be taken into account. When a certain solution for such a large-scale motion is obtained in a finite computational domain, and the major goal is to estimate the relevant sound emission in the far field (i.e., at the distances which are much longer than the characteristic size of that domain), then some try to do this by using a version of Kirchhoff's method, or through the application of Lighthill's acoustic analogy, although the latter should be regarded as a wrong way. If both sound waves and sound sources are to be determined just in the region with a high-unsteady vortical flow, this represents the most difficult problem, and generally this cannot be done without creating an adequate procedure of separating out the sound disturbances, as shown in section 4.

Thus, in the author's opinion, it is impossible to select an absolutely flawless method among the current approaches in computational aeroacoustics. Anyway, all those approaches do not enable one to resolve accurately both sound field and sound sources on the background of complex unsteady flow.

8.3. THE AUTHOR'S SOLUTIONS

In the past 30 years the author has accumulated substantial experience in fluid mechanics and aeroacoustics. The research works, both theoretical and computational, have been carried out in many important scientific directions. In that research a number of new approaches have been developed for the computational simulation of diverse non-steady flows with the analysis of intense heat transfer, mass forces and heat sources, hydrodynamic instability, evolution of both coherent structures and small-scale turbulence in jet flows, etc. A particular emphasis was placed on the investigation of aeroacoustic phenomena, including sound generation, non-linear sound–flow interactions in separated internal viscous flows, acoustic feedback and self-excited resonance effects. Most of these research efforts were aimed at designing new means of flow control which may be well applied to the solution of numerous practical problems. In the following only a part of all the research results will be enumerated.

A family of high-effective finite-difference schemes based on non-uniform grids, which could properly transform in the process of calculation (see, e.g., reference [64]), has been designed for integrating the Navier–Stokes or Euler equations with quite acceptable accuracy in the resolution of complex acoustic phenomena.

General approaches have been developed for the boundary control of unsteady subsonic flow in a computational domain. In comparison with the common way (solution of the initial-boundary-value problem under the fixed set of controlling parameters) these new approaches imply continuous analysis of the phenomena in the course of the solution (identification of coherent vortex structures, separating out the local regions with intense sound sources, etc.) along with successive use of diverse procedures of active control over the flow in order to obtain a desirable integral result, e.g., the reduction of total noise emission. Multipurpose models of permeable boundaries with controlled acoustic properties, including those with strong sound absorption [63–65, 68], have been created for the case of quasi-parallel mean flow near the boundary. The following important peculiarity has been revealed in the process of sound reflection from the boundary surface (with a specified distribution of the normal acoustic impedance) through which a normal subsonic flow takes place: the angle of incidence is not equal to the angle of reflection for a plane harmonic wave, and this difference depends on the local value of the Mach number for the normal component of the mean-flow velocity [117].

Very sophisticated procedures of active control, non-local in both time and space, have been first proposed in references [66, 67] to eliminate the spurious effects of transformation of vortex disturbances into acoustic waves on the outflow boundary (e.g., the exit section of a duct) through which a vortex-carrying subsonic flow escapes from the computational domain. With this aim two different methods have been designed. Within the first one the vortices are allowed to outflow freely through the boundary with minimum distortion in their structure, and hence with minimum sound emission. Following the second way, one should "wipe out" the intense vortices, which have reached the pre-exit zone, by applying the special procedure, so that no noise is emitted upstream during this procedure. Both these methods imply the continuous control of the flow evolution, at least in a certain pre-exit part of the computational domain. Thereby, radically new ways have been suggested for specifying the outflow boundary conditions in aeroacoustics, although these ways may be too complex in their realization (the procedures of flow control are able to take up the greater portion of computer program) to be widely applied in the routine practical calculations. Note that the numerous subsequent publications in computational fluid mechanics, including the most recent ones (see, e.g., references [29, 30, 32, 40, 118, 119]) do not expose the models of outflow boundaries which would be so effective and versatile as those given in reference [67].

All these methods were successfully applied to the investigation of various gas flows, both internal and free, and the most important solutions will be mentioned below.

Unsteady subsonic flows in an open rectangular cavity with self-generated acoustic oscillations were numerically investigated by integrating the two-dimensional Navier–Stokes equations for a compressible medium under steady boundary conditions [120, 121]. This was the first direct numerical simulation of the "whistle phenomenon" which is of extreme practical importance. Also the heat fluxes from hot gas to the cold wall, including the particular cases when the distributed blowing of cold gas through the cavity wall took place, and the effects of the external uniform longitudinal magnetic field imposed on the electro-conducting gas (it turned out that this means of flow control was able to suppress the self-excited acoustic oscillations in the cavity), etc., were studied in this class of viscous flows.

Computational investigation of non-steady viscous heat-conducting gas flows in ducts and nozzles of various shapes was carried out by using the Navier–Stokes equations. For instance, the non-linear acoustic oscillations, sometimes accompanied by unsteady separation, were studied in a finite cylindrical duct; these effects, as well as the general evolution of axisymmetric subsonic flow, were governed by unsteady boundary conditions (including those which provided fast suppression of acoustic oscillations) assigned at the inflow and outflow sections [64, 65].

As another example, unsteady axisymmetric flows in the domain containing the pre-nozzle chamber, Laval nozzle, and jet region behind the nozzle cutoff, were simulated (some results were presented in references [68, 122]). Acoustics in the chamber, shock triggering, upstream propagation of sound through the subsonic boundary layer, unsteady separation within the nozzle, instability of the detached supersonic jet, etc., were analyzed under active boundary control (changes in the nozzle shape, unsteady blowing and suction through permeable boundaries, the pressure variations specified in the ambient gas medium behind the nozzle cutoff). That was a true breakthrough in the long-standing problem of global simulation of unsteady viscous flows in nozzles, from subsonic flow in the pre-nozzle chamber and further to supersonic jet, and what is more, along with the analysis of accompanying non-linear acoustic processes. This success has been achieved primarily due to the radically new computational model which had been created to pose a consistent initial-boundary-value problem as well as to carry out the active boundary control of the flow evolution. Such a general computational model has removed many contradictions and mistakes which were inherent in the preceding approaches to this problem. Nevertheless, many subsequent works in this field were based on too primitive models.

Strong self-excited acoustic oscillations have been studied in the vicinity of the local subsonic zone close to the critical point, when a viscous supersonic jet, plane or round, impinges on a wall (1974–1977, unpublished).

The above solutions (most of them were obtained in the 1970s) have first displayed the surprising abilities of finite-difference methods to integrate the Navier–Stokes equations when applied to the solution of complex non-linear aeroacoustic problems, although previously this specific research direction was regarded as questionable. Besides, new ways have been found in designing the procedures of boundary control; moreover, it has been shown that application of such procedures, generally non-local in both time and space, is often a necessary demand for properly assigning the inflow and outflow boundary conditions in aeroacoustic initial-boundary-value problems.

A number of fundamental problems have been solved in theoretical and computational study of two-dimensional non-linear wave structures generated in subsonic and supersonic flows near a local zone of unsteady heat release due to the volume absorption of high-energy laser beam (see, e.g., reference [123]). An approximate theory of the process has been created, that enables one to predict the optimum relation between the Mach number of undisturbed flow and the characteristic level of heat release in order to provide the peak amplitude of the waves generated (strong shock waves in some cases).

A new mechanism of acoustic self-excitation was revealed in the course of numerical simulation of subsonic separated flows in suddenly expanded ducts [75, 76]. A fragment of flow picture from one of those solutions is shown in Figure 2, where the instantaneous vorticity field is presented in a suddenly expanded flat channel at $\mathrm{Re} = \rho_0 h u_0/\mu_0 \approx 1000$, $M_0 \approx M_{\max} \approx u_0/a_0 \approx 0{\cdot}4$. The near-uniform core of the inlet mean-velocity profile $U(y)$, assigned at $x = -0{\cdot}5$, has the width $h_c \approx 0{\cdot}6h$, where h is the width of the narrow part of the channel (at $x < 0$), and the characteristic longitudinal velocity u_0 is taken closely to the center point of the profile $U(y)$. Note that this asymmetric flow took place under completely symmetric (in relation to the channel axis $y = 0$) boundary conditions.

This flow is characterized by very complex, even stochastic, dynamics, with self-generated acoustic oscillations in the whole volume. The successive shedding of vortex perturbations from the edges at $x = 0$, as well as the further formation of intense vortices, caused the related system of sound sources which in turn supported the dominant transverse acoustic

Figure 2. Unsteady subsonic two-dimensional flow in the channel with sudden expansion.

oscillations in the wide part of the channel (the width $L = 3h$) with the dimensionless wavelength $\lambda_n/h \approx 6/n$, $n = 1, 3, 5, \ldots$. These computational results were supplemented by the theoretical analysis, and the necessary conditions of a self-excited resonance have been derived for expanded ducts of various shapes. Some high-efficient means of active and passive boundary control were suggested and tested to avoid such resonance effects. Besides, diverse versions of this specific problem served as an experimental range for testing the most sophisticated computational approaches which were developed and first applied by the author (e.g., the non-local procedures of active control aimed at specifying the outflow boundary conditions [67]). This solution has been chosen as a visual example because this flow is too inconvenient for application of all the traditional theoretical models of sound generation. Indeed, here the whole domain is filled with numerous sound sources, and so the acoustic compenents should be clearly separated out from the unsteady flow variables just in the region of sound generation if one wishes to understand the mechanism of acoustic self-excitation. Note that no farfield approximation is applicable to this case, as well as to most of the internal flows.

A new computational method [83] has been designed for the simulation of unsteady subsonic turbulent jet flows. Within this method the non-steady Reynolds equations for large-scale vortical motion were simultaneously coupled with multiparameter models of small-scale turbulence. The most general version of such a model contained the three independent scalar variables for which the evolutionary equations were written: the energy of turbulent motion $e = \rho\langle(u_1')^2 + (u_2')^2 + (u_3')^2\rangle/2$, the shear stress $\tau = -\rho\langle u_1' u_2'\rangle$, and the characteristic frequency ω (i.e., in the two-dimensional case it was assumed that $\langle u_1' u_3'\rangle$ and $\langle u_2' u_3'\rangle$ were negligible, but $\langle(u_3')^2\rangle \neq 0$); the two-parameter versions were applied as well. The universal coefficients in those equations were derived from the set of well-studied turbulent flows. The important feature of this method, which seems to be most promising in jet problems, should be emphasized: in this way many complex acoustic phenomena can be also simulated (surely when the sound wavelength is much longer than the characteristic spatial scale of turbulence), including the processes of sound generation due to evolution of large-scale vortices. Therefore, that work was the first example of such a general approach applied to the study of non-linear interactions between the large-scale vortical motion and the fine-grained turbulence in its downstream evolution, and moreover, to the quest for new means of control over these interactions and the related sound emission. Although this approach was applied by the author only to the simulation of two-dimensional flows, it is general enough to be extended to a great number of three-dimensional problems. With the use of this method the non-linear interactions between coherent structures and small-scale turbulence were investigated in the plane mixing layers with harmonic excitation at the inlet section. Under the definite control conditions (when the specific structure of turbulence was assigned at the inlet section) the pronounced suppression of large vortices growth by small-scale turbulence occurred, and in turn a considerable decrease in coherent sound emission from the layer was obtained.

The unique set of all these computational solutions gave great impulse to the author's work on the development of new theoretical approaches in aeroacoustics. It turned out that the conventional aeroacoustic models were too imperfect to supplement adequately the above computational research, especially in analyzing the sound sources in high-unsteady flows.

9. CONCLUSIONS

A number of basic theoretical approaches in aeroacoustics have been considered, and particular attention has been focused on diverse models of volume sound sources which can be assigned externally or arise due to high-unsteady flow structure. Though the main conclusions were given in each section above, the most important among them can be summarized as follows.

On analyzing the ordinary linear systems of acoustic equations without source terms, one may come to the not very comforting conclusion that the only model which can simulate the evolution of acoustic disturbances in the most correct manner is the version of Blokhintsev's equations for *uniform steady* mean flow. In more general cases these systems are able to describe the small disturbances of all kinds (i.e., the fluctuations of both vorticity and entropy as well as the acoustic waves), but it is extremely difficult to find an accurate method for separating out the net sound from all these disturbances which often seem to be intricately interconnected. A lot of known solutions to linear acoustic problems (e.g., sound propagation in ducts with steady background flows) are obtained only due to rather coarse assumptions on the form of the prospective solution, and quite general initial-boundary-value problems are very rarely found among scientific publications.

An adequate decomposition of the vector $\mathbf{Z}(\mathbf{r}, t) = \{u_1, u_2, u_3, s, p, \rho\}$ into the sound disturbances $\mathbf{Z}_\alpha(\mathbf{r}, t)$ and the variables of unsteady background flow $\mathbf{Z}_v(\mathbf{r}, t)$ (i.e., the latter corresponds to non-acoustic motion) has been formulated in section 4 as the key problem in aeroacoustics. The long absence of such a general approach seems to be the main reason for all unsuccessful attempts to define the aerodynamic sound sources. It has been shown that a closed system of non-linear equations governing unsteady background flow (that represents a specific medium where all sound waves have to be characteristically precluded) should be first derived; then the solution $\mathbf{Z}_v(\mathbf{r}, t)$ of a certain initial-boundary-value problem could be taken as a known function while deriving a relevant closed system of non-linear acoustic equations in which the aerodynamic sound source $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, q_s\}$ would be logically defined as dependent on that background-flow solution. So this approach changes radically the traditional concept of sound–flow interactions in inviscid gas media.

A certain procedure of time averaging is frequently used in efforts to obtain the system of equations governing the evolution of small disturbances on the background of average flow. If one considers a typical example of its application to an aeroacoustic problem, all details of this procedure are not discussed as usual, but it has been shown above that any changes in the values of a few decisive parameters may lead to drastic consequences in the final solution. In fact, after applying the integral operator of time averaging to the basic *local* system of non-linear partial differential equations of fluid mechanics one will obtain a much more complex *non-local* system of integro-differential equations which demands special methods of solution. Though generally such a procedure represents a powerful method applicable to the study of many physical processes, it cannot serve as a universal means for an adequate decomposition of any flow into acoustic and nonacoustic fields, and an additional number of coarse assumptions is usually required to provide the closure of the problem.

Among the alternative approaches the decomposition of the velocity field into solenoidal and potential parts looks quite promising, but this procedure should be accompanied by properly splitting all other variables. The initial and boundary conditions must be decomposed as well, but often this can be implemented in different ways that may introduce a non-uniqueness. Besides, in the general case it seems hardly probable to prove that the solenoidal component of the velocity is related exactly to the mean flow, and the potential component is connected with sound. Nevertheless, an accurate procedure of this kind was applied in section 4.4 to irrotational homentropic flow, and that example displayed the presence of volume sound sources which were determined by the previously obtained solution of the relevant elliptic problem for the irrotational unsteady background flow. This important result refutes the universally adopted opinion that only a flow region with non-zero vorticity can yield volume sound sources. At the same time no one can deny that the evolution of high-vortical structures, and their instability in particular, is often the main reason for sound emission.

The traditional concept of the externally assigned source terms in aeroacoustics (those, being "small", act only on the sound field) is evidently wrong. The key question remains unclear regarding what part of the source action should be attributed to the sound field, and which other one will influence the unsteady background flow. Generally, the notion of an externally assigned source is rather ambiguous in aeroacoustics, and so it should be supplemented with some additional requirements on the source functions $\{\mathbf{F}, \mu, q\}$; otherwise many mistakes may occur in applying the relevant equations, as well as in analyzing their characteristic properties. A few examples have been given in section 5 to show that existing acoustic models with external source terms, both linear and non-linear, do not work (e.g., the particular case when $\nabla\mathbf{F} = 0$). This sizable gap in theoretical aeroacoustics creates many obstacles in the solution of numerous practical problems (aerothermoacoustic phenomena due to volume heat release, effects of electromagnetic or gravitational forces, and so on).

Diverse approaches have been developed for the theory of sound sources which may arise due to high-unsteady fluid motion. There a lot of manipulations have been made with the basic non-linear equations of fluid mechanics in attempts to derive an appropriate mathematical model of aerodynamic sound. All mathematical transformations could be absolutely correct at each step, but a final second order scalar equation (e.g., equations (45),(62),(64)) did not offer the best means for the *separate* study of sound waves. By the way, each of these scalar "acoustic" equations is not closed since it contains all the set of unknown *total* variables $\{\mathbf{u}, s, p, \rho\}$. No rigorous mathematical proofs have been given to justify that a certain group of the second-order differential terms on the right-hand side of such an equation can be treated as a "sound source". Clearly, the general non-linear systems of Navier–Stokes and Euler equations for a compressible medium are quite sufficient to simulate a lot of complex phenomena without any decomposition of total variables $\{\mathbf{u}, s, p, \rho\}$. But one urgently needs specific mathematical models in which the acoustic processes, especially the non-linear effects of sound generation by flow, can be accurately separated out.

Concerning Lighthill's acoustic analogy, one has to say that the famous equation (45) by no means can serve as an adequate basis for the definition of aerodynamic sound sources, and all attempts to use it for such a definition are featured by gross mathematical errors. This model is based on the false illusion that an appropriate *scalar* second order equation should be derived from equations (1), (2), and a certain group of terms in that equation may yield an exact formula for the sound source Q_L. Note that *the only scalar source* is there supposed to be sufficient; thereby the general concept of sound sources, in which these latter must be defined by the vector $\mathbf{Y}_s = \{\mathbf{F}_s, \mu_s, q_s\}$, is ignored. The assumption of the source compactness cannot save this situation. Moreover, equation (45) is not related directly to the problem of sound sources, and generally to acoustics, and with equal success it may be derived even within the classical model of incompressible fluid flow, as shown in section 6. "Alternative equations" (54) and (55) are the most striking examples illustrating the absurd concept of this approach. Thus, Lighthill's acoustic analogy should be recognized as one of the greatest delusions in fluid mechanics. A most surprising fact should be mentioned: it seems that no one has tried to reveal the true essence of this delusion during the quite long period since paper [45] was published.

Most subsequent approaches were influenced greatly by Lighthill's acoustic analogy, and so all those have inherited its principal defects. For instance, "the most general" equation

(45) has been modified and simplified with the aim of obtaining an approximate expression for the "sound source" Q_L in subsonic flows. Then both right- and left-hand sides of equation (45) have been considered *separately* (that was most mistaken), and typically some "small terms" omitted as negligible; as a result, the only merit of equation (45), its exactness, has been lost. In this way, the Ribner and Powell formulations for a homentropic flow have been obtained. In the latter the sound source was assumed to be determined by a pre-found solution to the "similar" flow of incompressible fluid, and the main conclusion was expressed (unfortunately, it had given rise to serious consequences in further aeroacoustic research) that only a flow region with non-zero vorticity could contribute to sound emission. Each model of this kind shows new weak points in addition to those in the basic model [45]. So the known set of "modified" models offered for the sound generation in unsteady subsonic flows is inadequate as well.

Within other approaches (e.g., those given by Howe, Phillips and Lilley) diverse forms of an exact *scalar* second order equation have been suggested as versions of a "generalized inhomogeneous convected wave equation". In a wide sense, these attempts followed Lighthill's concept in which a scalar second order equation was supposed to serve as a basis for defining *the only scalar* sound source. All the same, the right-hand sides of these equations cannot be identified as sound sources.

Dealing with such a great variety of known approaches to the theory of aerodynamic sound sources, and even without analyzing these in detail, one may feel some doubts whether such different models can be applied with equal success. Replying to this delicate question, Ffowcs-Williams [124] asserted that "the nature of aeroacoustic fields permits many different but *equally exact* computational procedures for evaluating both the sound and its source field". This explanation looks quite sufficient and convenient for many, but it does not contribute to the search for the truth.

One can also try to characterize the accumulated bulk of results in both experimental and computational study of aerodynamic phenomena with sound generation. Surely, all this experience is of great value, but it looks as if it has not contributed much to the process of impartially analyzing and revising the old theories. The inability to measure directly all components of the sound source $\mathbf{Y}_s$ at each point of a real high-unsteady flow seems to be the main, and probably unavoidable, obstacle for this natural process. As for the numerical simulation, this research direction is completely dependent on the whole mathematical model applied, including basic equations, boundary models, finite-difference scheme, possible means of flow control, etc. So all the available collection of computational methods forms a separate theoretical basis for such a research, but most of these methods are imperfect to a greater or lesser extent. At first sight it may seem that in applying the DNSC-approach one needs no definition of sound sources caused by high-unsteady flow structure, but in analyzing a solution obtained within this approach, and especially in trying to distinguish the key acoustic effects from non-acoustic motion, one cannot do without an adequate theory of aerodynamic sound. Clearly, a complete set of efficient theoretical approaches must support any computational or experimental investigation of a definite flow, if the major research purpose is to understand the tangled mechanism of sound generation (and to find means of control over these phenomena), not just to estimate solely the approximate noise level in the far field.

Thus, many inherent flaws, illusions and delusions have been found in the theoretical fundamentals of aeroacoustics, and especially in the general approaches to the theory of aerodynamic sound sources. The author hopes that this critical analysis will be helpful in developing new, much more accurate, theoretical models in this important scientific field.

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